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# DYNAMIC RAYS OF BOUNDED-TYPE TRANSCENDENTAL SELF-MAPS OF THE PUNCTURED PLANE

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Abstract. We study the escaping set of functions in the class $\mathcal{B}^*$, that is, transcendental self-maps of $\mathbb{C}^*$ for which the set of singular values is contained in a compact annulus of $\mathbb{C}^*$ that separates zero from infinity. For functions in the class $\mathcal{B}^*$, escaping points lie in their Julia set. If $f$ is a composition of finite order transcendental self-maps of $\mathbb{C}^*$ (and hence, in the class $\mathcal{B}^*$), then we show that every escaping point of $f$ can be connected to one of the essential singularities by a curve of points that escape uniformly. Moreover, for every sequence $e \in \{0,\infty\}^{\mathbb{N}_0}$, we show that the escaping set of $f$ contains a Cantor bouquet of curves that accumulate to the set $\{0,\infty\}$ according to $e$ under iteration by $f$.

## 1. **Introduction.**

Complex dynamics concerns the iteration of a holomorphic function on a Riemann surface $S$. Given a point $z \in S$, we consider the sequence given by its iterates $f^n(z) = (f \circ \overset{n}{\cdots} \circ f)(z)$ and study the possible behaviours as $n$ tends to infinity. We partition $S$ into the *Fatou set*, or stable set,

$$F(f) := \left\{ z \in S \ : \ (f^n)_{n\in\mathbb{N}} \text{ is a normal family in some neighbourhood of } z \right\}$$

and the *Julia set*, or chaotic set, $J(f) := S \setminus F(f)$. If $f : S \subseteq \hat{\mathbb{C}} \to S$ is holomorphic and $\hat{\mathbb{C}} \setminus S$ consists of essential singularities, then conjugating by a Möbius transformation, we can reduce to the following three cases:

- $S = \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and $f$ is a rational map;
- $S = \mathbb{C}$ and $f$ is a transcendental entire function;
- $S = \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and *both* zero and infinity are essential singularities.

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We study this third class of maps, which we call *transcendental self-maps of* $\mathbb{C}^*$. Such maps are all of the form

$$f(z) = z^n \exp\big(g(z) + h(1/z)\big), \tag{1}$$

where $n \in \mathbb{Z}$ and $g, h$ are non-constant entire functions. The number $n \in \mathbb{Z}$ is called the *index* of $f$, written $\operatorname{ind}(f) = n$, and equals the index (or winding number) of the curve $f(\gamma)$ with respect to the origin for any positively oriented simple closed curve $\gamma$ around the origin. Transcendental self-maps of $\mathbb{C}^*$ arise in a natural way in many instances, for example, when you complexify circle maps, like the so-called Arnol'd standard family: $f_{\alpha\beta}(z) = ze^{i\alpha}e^{\beta(z-1/z)/2}$ for $0 \leqslant \alpha \leqslant 2\pi,\ \beta \geqslant 0$ [18] (see Figure 1). Note that if $f$ has three or more omitted points, then, by Picard's theorem, $f$ is constant and, consequently, a non-constant holomorphic function $f : \mathbb{C}^* \to \mathbb{C}^*$ has no omitted values. The text [33] is a basic reference on the iteration of holomorphic functions in one complex variable. See [5] for a survey on the iteration of transcendental entire and meromorphic functions.

Although the iteration of transcendental (entire) functions dates back to the time of Fatou [19], Rådström [36] was the first to consider the iteration of holomorphic self-maps of $\mathbb{C}^*$. An extensive list of references on this topic can be found in [29]. It is our goal in this paper to continue with the program started in [30] of a systematic study of holomorphic self-maps of $\mathbb{C}^*$, extending the modern theory of iteration of transcendental entire functions to this setting.

To that end, we recall the definition of the *escaping set* of an entire function $f$,

$$I(f) := \{z \in \mathbb{C} \ :\ f^n(z) \to \infty \text{ as } n \to \infty\},$$

whose investigation has provided important insight into the Julia set of entire maps. For polynomials, the escaping set consists of the basin of attraction of infinity and its boundary equals the Julia set. For transcendental entire functions, Eremenko showed that $I(f) \cap J(f) \neq \emptyset$, $J(f) = \partial I(f)$ and the components of $\overline{I(f)}$ are all unbounded [16]. Similar properties [30, Theorems 1.1, 1.3 and 1.4] hold for transcendental self-maps of $\mathbb{C}^*$ once the definition is adapted to take both essential singularities into account. More precisely, the *escaping set* of a transcendental self-map of $\mathbb{C}^*$ is given by

$$I(f) := \{z \in \mathbb{C}^* \ :\ \omega(z, f) \subseteq \{0, \infty\}\},$$

where $\omega(z, f)$ is the classical omega-limit set and the closure is taken in $\hat{\mathbb{C}}$,

$$\omega(z, f) := \bigcap_{n \in \mathbb{N}} \overline{\{f^k(z) \ :\ k \geqslant n\}}.$$

As usual, the set of singularities of the inverse function $\operatorname{sing}(f^{-1})$, which consists of the critical values and the finite asymptotic values of $f$, plays an important role. In the entire setting, the so-called *Eremenko-Lyubich class*

$$\mathcal{B} := \{f \text{ transcendental entire function } :\ \operatorname{sing}(f^{-1}) \text{ is bounded}\}$$

consisting of *bounded-type* functions was introduced in [17] (see also [47]). Eremenko and Lyubich showed that if $f \in \mathcal{B}$, then $I(f) \subseteq J(f)$ or, in other words, the Fatou set has no escaping components. Functions in the class $\mathcal{B}$ have many other useful properties; see, for example, [44, 32, 4]. In the context of transcendental self-maps of $\mathbb{C}^*$, the analogous class to consider is that where the singular values stay away from both essential singularities, hence we introduce the class of *bounded-type* transcendental self-maps of the punctured plane as

$$\mathcal{B}^* := \{f \text{ transcendental self-map of } \mathbb{C}^* \ : \ \operatorname{sing}(f^{-1}) \text{ is bounded away from } 0, \infty\}$$

and prove the following result.

**Theorem 1.1.** *Let $f \in \mathcal{B}^*$. Then $I(f) \subseteq J(f)$.*

As shown in [31], functions outside the class $\mathcal{B}^*$ may have escaping Fatou components: either *Baker domains*, which are periodic Fatou components in $I(f)$, or *wandering domains*, which are Fatou components $U$ such that $f^m(U) \cap f^n(U) \neq \emptyset$ implies $m = n$. It remains an open question whether functions in the class $\mathcal{B}^*$ can have wandering domains (outside the escaping set), as it is the case for entire functions in the class $\mathcal{B}$ [10, Theorem 17.1].

It is a natural question to investigate the relationship between entire functions in the class $\mathcal{B}$ and self-maps of $\mathbb{C}^*$ in the class $\mathcal{B}^*$. Keen [24] showed that if $g$ and $h$ are polynomials and $n \in \mathbb{Z}$, then the function $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$ has a finite number of singular values and hence belongs to the class $\mathcal{B}^*$. The next theorem extends this results to all functions in the class $\mathcal{B}$ when $n = 0$.

**Theorem 1.2.** *Let $g$ and $h$ be entire functions in the class $\mathcal{B}$. Then the function $f(z) = \exp\big(g(z) + h(1/z)\big)$ is in the class $\mathcal{B}^*$.*

As opposed to the situation for entire functions, there is a deep relation between the bounded-type condition for holomorphic self-maps of $\mathbb{C}^*$ and their order of growth. To be more precise, recall that the *order* and *lower order* of an entire function $f$ can be defined, respectively, as

$$\rho(f) := \limsup_{r \to +\infty} \frac{\log \log M(r, f)}{\log r} \quad \text{and} \quad \lambda(f) := \liminf_{r \to +\infty} \frac{\log \log M(r, f)}{\log r},$$

where $M(r, f) := \max_{|z|=r} |f(z)|$. If $f$ is a transcendental self-map of $\mathbb{C}^*$, then we also need to take into account the essential singularity at zero. Hence the *order* of growth is given by the two quantities

$$\rho_\infty(f) := \limsup_{r \to +\infty} \frac{\log \log M(r, f)}{\log r} \quad \text{and} \quad \rho_0(f) := \limsup_{r \to 0} \frac{\log \log 1/m(r, f)}{\log 1/r},$$

where $M(r, f)$ is as before and $m(r, f) := \min_{|z|=r} |f(z)|$. We say that $f$ has *finite order* if both $\rho_\infty(f) < +\infty$ and $\rho_0(f) < +\infty$. Likewise, we can define two quantities associated with the lower order of such functions, $\lambda_\infty(f)$ and $\lambda_0(f)$, by replacing $\limsup$ by $\liminf$ in the expressions above. An important property of entire functions $f \in \mathcal{B}$ is that $\lambda(f) \geqslant 1/2$ [7, 27] (see also [42, Lemma 3.5]). The next result shows that, surprisingly, the lower order of a function in $\mathbb{C}^*$ always equals its order and hence it is not relevant in this setting. Moreover, if the order is finite, then it is an integer.

**Theorem 1.3.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$. Then $\lambda_0(f) = \rho_0(f)$ and $\lambda_\infty(f) = \rho_\infty(f)$. If $f$ has finite order, then $f(z) = z^n \exp\big(P(z) + Q(1/z)\big)$, where $n \in \mathbb{Z}$ and $P, Q$ are polynomials, and therefore $\rho_0(f), \rho_\infty(f) \in \mathbb{Z}$ and $f$ has finite type. In particular, transcendental self-maps of $\mathbb{C}^*$ of finite order are in the class $\mathcal{B}^*$ and $\lambda_0(f), \lambda_\infty(f) \geqslant 1$.*

In [16], Eremenko conjectured that if $f$ is a transcendental entire function, then the components of $I(f)$ are all unbounded. A stronger version of this conjecture states that every escaping point can be joined to infinity by a curve of points that escape uniformly. Such curves are called *ray tails* and their maximal extensions are called *dynamic rays*. Douady and Hubbard [15] were the first to introduce dynamic rays to study the dynamics of polynomials, where $I(f)$ consists of the attracting basin of infinity which is connected. Devaney and Krych [13] showed that for maps in the exponential family $E_\lambda(z) = \lambda e^z$, $\lambda \in (0, 1/e)$, the Julia set consists of dynamic rays (that they called *hairs*). Devaney and Tangerman [14] proved that the same holds for certain *finite-type* functions, that is, functions with finitely many singular values, satisfying additional technical conditions, such as the sine family $S_\lambda(z) = \lambda \sin(z)$, $\lambda \in (0, 1)$. They coined the term *Cantor bouquet* to describe the Julia set of these functions. They first defined a Cantor $N$-bouquet, where $N \in \mathbb{N}$, to be a subset of $J(f)$ homeomorphic to the product of a Cantor set and the half-line $[0, +\infty)$, and then a Cantor bouquet to be an increasing union of Cantor $N$-bouquets. However, this is somewhat different to the definition of Cantor bouquet used more recently (and in this paper) in terms of a topological object called a *straight brush* which is due to Aarts and Oversteegen [1] (see Definition 9.1).

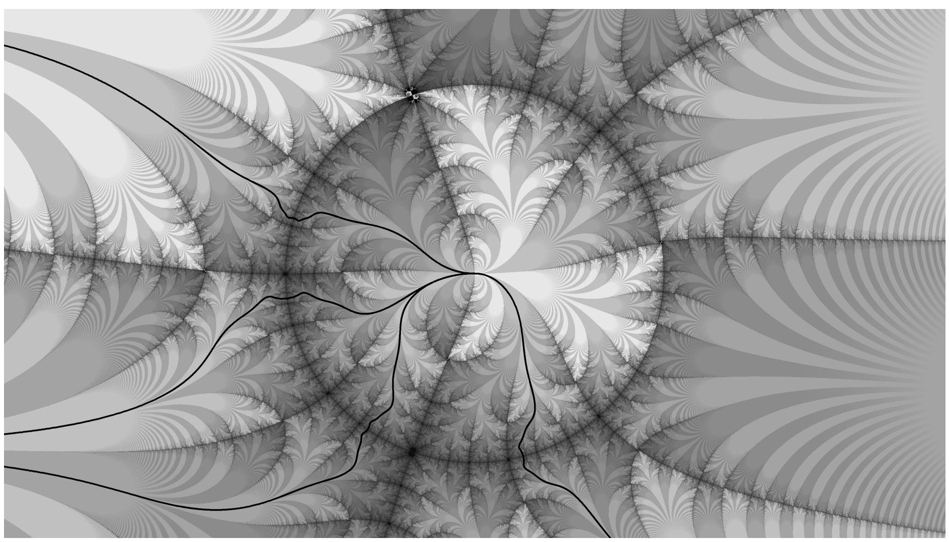

FIGURE 1. Period 8 cycle of rays landing on a repelling period 4 orbit in the unit circle for the function $f_{\alpha\beta}(z) = ze^{i\alpha}e^{\beta(z-1/z)/2}$ from the Arnol'd standard family, with $\alpha = 0.19725$ and $\beta = 0.48348$. Such points lie in the set $I_e(f_{\alpha\beta})$ with $e = \overline{\infty 0 \infty \infty 0 \infty 00}$ (see (2)).

Rottenfußer, Rückert, Rempe and Schleicher proved in [44, Theorem 1.2] that the stronger version of Eremenko's conjecture holds for transcendental entire functions of bounded type and finite order or, more generally, a finite composition of such functions: every escaping point can be joined to infinity by a curve of points that escape uniformly. This result was proved independently by Barański [2, Theorem C] for *disjoint-type* functions, that is, transcendental entire functions for which the Fatou set consists of a completely invariant component which is a basin of attraction. Shortly after, Barański, Jarque and Rempe proved that, actually, the Julia set of the functions considered in [44] contains a Cantor bouquet [3, Theorem 1.6].

In this article we prove the existence of dynamic rays for transcendental self-maps of $\mathbb{C}^*$ by adapting the construction of [44] to our setting. We use the notation $f^n_{|\gamma} \to \{0,\infty\}$ to mean that, under iteration by $f$, the points in $\gamma$ escape to zero, escape to infinity or accumulate to both of them and nowhere else.

**Theorem 1.4.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ of finite order or, more generally, a finite composition of such functions. Then every point $z \in I(f)$ can be connected to either zero or infinity by a curve $\gamma$ such that $f^n_{|\gamma} \to \{0,\infty\}$ uniformly in the spherical metric.*

Note that in the statement of Theorem 1.4 there is no assumption of bounded-type. This is because, as we mentioned above, finite order transcendental self-maps of $\mathbb{C}^*$ are always in the class $\mathcal{B}^*$ (see Lemma 4.6).

Given a holomorphic self-map of $\mathbb{C}^*$, a *lift* of $f$ is an entire function $\tilde{f}$ satisfying $\exp \circ \tilde{f} = f \circ \exp$. Bergweiler [6] proved that $J(\tilde{f}) = \exp^{-1} J(f)$. Seeing this result one might think that every result about entire functions could be extended to self-maps of $\mathbb{C}^*$ via their lifts. Unfortunately, this is not possible. In particular, a lift of a map of bounded type is never of bounded type, its singular set is contained in a vertical band and so, we cannot apply directly the results from [44]. In fact, in the opposite direction, Theorem 1.4 allows us to construct dynamic rays for some transcendental entire functions that are not in the class $\mathcal{B}$, but project to functions in the class $\mathcal{B}^*$ satisfying the hypothesis of Theorem 1.4.

**Corollary 1.5.** *Let $f$ be a transcendental entire function of the form*

$$f(z) = nz + P(e^z) + Q(e^{-z}),$$

*with $n \in \mathbb{Z}$ and $P, Q$ polynomials, or a finite composition of such functions. Then every point $z \in I(f)$ with $|\operatorname{Re} f^n(z)| \to +\infty$ as $n \to \infty$ can be connected to infinity by a curve of points that escape uniformly.*

The main tool to prove Theorem 1.4 is the use of logarithmic coordinates, introduced by Eremenko and Lyubich [17], and the expansivity of the logarithmic transform near the essential singularities. The orbit of escaping points eventually enters the tracts (unbounded Jordan domains which are mapped to a neighbourhood of zero or infinity) and remains there. We partition each tract into fundamental domains, each with a corresponding symbol, and consider itineraries on them; see Section 5 for the precise definitions. Observe that the previous theorem contains no claim of which dynamic rays actually exist. Our next result shows that, under the hypothesis of Theorem 1.4, there is a unique dynamic ray for every sequence of fundamental domains that contains only finitely many symbols. Here $P(f)$ denotes the *postsingular set* of $f$ which is the closure of the union of all the (forward) iterates of $\operatorname{sing}(f^{-1})$. We say that a dynamic ray $\gamma$ *lands* if $\overline{\gamma} \setminus \gamma$ is a single point.

**Theorem 1.6.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ of finite order or, more generally, a finite composition of such functions, and let $\underline{t} = (D_n)$ be an admissible sequence of fundamental domains of $f$ containing only finitely many symbols. Then the function $f$ has a unique non-empty dynamic ray $\gamma$ with external address $\underline{t}$. Furthermore, if $\underline{t}$ is periodic and the set $P(f)$ is bounded in $\mathbb{C}^*$, then the dynamic ray $\gamma$ lands.*

Observe that, for example, Theorem 1.6 implies that every fundamental domain contains exactly one fixed dynamic ray.

We associate to each escaping point an *essential itinerary* $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ defined by

$$e_n := \begin{cases} 0, & \text{if } |f^n(z)| \leqslant 1, \\ \infty, & \text{if } |f^n(z)| > 1, \end{cases} \tag{2}$$

for all $n \in \mathbb{N}_0$. We use the notation $\overline{e_0 e_1 \ldots e_{n-1}}$ to denote the $n$-periodic sequence which consists of $e_0 e_1 \ldots e_{n-1}$ repeated infinitely often. Consider, for each sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$, the set of points whose essential itinerary is eventually a shift of $e$, that is,

$$I_e(f) := \{z \in I(f) \ : \ \exists \ell, k \in \mathbb{N}_0, \ \forall n \in \mathbb{N}_0, \ |f^{n+\ell}(z)| > 1 \Leftrightarrow e_{n+k} = \infty\}.$$

Each of the sets $I_e(f)$, $e \in \{0, \infty\}^{\mathbb{N}_0}$, should be regarded as the analogue of the whole of $I(f)$ for a transcendental entire function $f$. In [30, Theorem 1.1] it is shown that, for each $e \in \{0, \infty\}^{\mathbb{N}_0}$, $I_e(f) \cap J(f) \neq \emptyset$. We follow the methods of [3] to show that, in fact, under the hypothesis of Theorem 1.4, each set $I_e(f)$ not only contains periodic ray tails (countably many) but a Cantor bouquet.

**Theorem 1.7.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ of finite order or, more generally, a finite composition of such functions. For each $e \in \{0, \infty\}^{\mathbb{N}}_0$, there exists a Cantor bouquet $X_e \subseteq I_e(f)$ and, in particular, the set $I_e(f)$ contains uncountably many ray tails.*

Although Theorem 1.4 is stated in terms of functions of finite order, its proof is more general and applies to a class of functions satisfying certain *good geometry properties* (see Definition 3.13). Rempe, Rippon and Stallard showed that, assuming an extra condition (namely, that the tracts have what they call *bounded gulfs*), the ray tails constructed in [44] consist of fast escaping points [40, Theorem 1.2]. It seems likely that similar conditions would imply that the dynamic rays that we construct here are also fast escaping in the sense of [30].

**Remark 1.8.** Lasse Rempe-Gillen pointed out that Theorem 1.4 may also be proved using random iteration as described in the last paragraph of [44, Section 5] by taking, for $R > 0$ sufficiently large,

$$f_1(z) := \begin{cases} f(z) & \text{if } |f(z)| > R, \\ 1/f(z) & \text{if } |f(z)| < 1/R; \end{cases} \qquad f_2(z) := \begin{cases} f(1/z) & \text{if } |f(1/z)| > R, \\ 1/f(1/z) & \text{if } |f(1/z)| < 1/R; \end{cases}$$

which both have a logarithmic transform in the class $\mathcal{B}_{log}$ and then applying the results of [44] to a non-autonomous sequence of these two functions. However, it seems natural to provide a direct proof.

**Structure of the paper.** Roughly speaking, the first half of the paper is devoted to describing the basic properties of functions in the class $\mathcal{B}^*$ and in the second half we investigate the existence of dynamic rays for these functions. In Section 2, we study what is the relation between the classes $\mathcal{B}$ and $\mathcal{B}^*$; the proof of Theorem 1.2 is there. In Section 3, we describe the geometry of logarithmic coordinates of functions in the class $\mathcal{B}^*$ and give the proof of Theorem 1.1. Finite order functions are introduced in Section 4, where we prove Theorem 1.3, and are shown to be examples of functions with good geometry. In Section 5, we define symbolic dynamics, both in terms of essential itineraries (with respect to essential singularities) and external addresses (with respect to tracts). In contrast to what happens in the entire case, in our setting the Bernoulli shift map is a subshift of finite type, where only some sequences are admissible. In Section 6, we show that if an external address $\underline{s}$ is periodic, then the set $J_{\underline{s}}(F)$ consisting of all points with that address contains an unbounded continuum of fast escaping points, this is used later to prove Theorem 1.6 in Section 9. Dynamic rays are introduced in Section 7. Finally, the proofs of Theorem 1.4 and Theorem 1.7 are sketched in Section 8 and Section 9, respectively, focusing on the differences with the proofs of [44, Theorem 1.2] and [3, Theorem 1.6], which concern entire functions.

**Notation.** In this paper $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. If $z \in \mathbb{C}^*$ and $X \subseteq \mathbb{C}^*$, then $\operatorname{dist}(z, X)$ denotes the Euclidean distance from $z$ to $X$. If $X, Y$ are disjoint sets, we use $X \sqcup Y$ to denote the union of $X$ and $Y$. If $z_0 \in \mathbb{C}$ and $0 < r < r'$, we define the sets

$$D(z_0, r) := \{z \in \mathbb{C} \ : \ |z - z_0| < r\}, \quad A(r, r') := \{z \in \mathbb{C} \ : \ r < |z| < r'\}.$$

We define the half-planes $\mathbb{H}^+ := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > 0\}$, $\mathbb{H}^- := \{z \in \mathbb{C} \ : \ \operatorname{Re} z < 0\}$ and, for $r \in \mathbb{R}$, we put

$$\mathbb{H}^+_r := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > r\}, \quad \mathbb{H}^-_r := \{z \in \mathbb{C} \ : \ \operatorname{Re} z < -r\},$$

and, for $r > 0$, $\mathbb{H}^\pm_r := \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| > r\} = \mathbb{H}^+_r \sqcup \mathbb{H}^-_r$. If $X$ is a set in $\mathbb{C}^*$, then the topological operations $\overline{X}$ and $\partial X$ are taken in $\mathbb{C}^*$ unless stated otherwise, and we use $\widehat{X}$ to denote the closure of $X$ in $\hat{\mathbb{C}}$.

2. **Functions in the class $\mathcal{B}^*$.** Let $f$ be a transcendental entire function or a transcendental self-map of $\mathbb{C}^*$. We say that $v \in \hat{\mathbb{C}}$ is a *critical value* of $f$ if $v = f(c)$ with $f'(c) = 0$. We say that $a \in \hat{\mathbb{C}}$ is an *asymptotic value* of $f$ if there is a continuous injective curve $\gamma : (0, +\infty) \longrightarrow \hat{\mathbb{C}}$, an *asymptotic path*, such that $\gamma(t) \to \alpha$ as $t \to +\infty$, where $\alpha$ is an essential singularity of $f$, and $f(\gamma(t)) \to a$ as $t \to +\infty$. Let $\operatorname{CP}(f)$ denote the set of critical point of $f$. The set of singularities of the inverse function, $\operatorname{sing}(f^{-1})$, consists of the critical values of $f$, $\operatorname{CV}(f) := f(\operatorname{CP}(f))$, and the finite asymptotic values of $f$, $\operatorname{AV}(f)$, that is

$$\operatorname{sing}(f^{-1}) = \operatorname{CV}(f) \cup \operatorname{AV}(f).$$

In $\mathbb{C}^*$, by finite asymptotic value we mean that $a \notin \{0, \infty\}$. For transcendental self-maps of $\mathbb{C}^*$, we can decompose $\operatorname{AV}(f)$ as

$$\operatorname{AV}(f) = \operatorname{AV}_0(f) \cup \operatorname{AV}_\infty(f),$$

depending on whether $a \in \operatorname{AV}(f)$ has an asymptotic path $\gamma$ to zero or to infinity. Observe that the set $\operatorname{AV}_0(f) \cap \operatorname{AV}_\infty(f)$ may be non-empty. Finally, we define the *singular set* of $f$, $S(f)$, and the *postsingular set* of $f$, $P(f)$, as

$$S(f) := \overline{\operatorname{sing}(f^{-1})} \quad \text{and} \quad P(f) := \overline{\bigcup_{n \in \mathbb{N}} f^n\big(\operatorname{sing}(f^{-1})\big)}.$$

Note that $S(f)$ contains the accumulation points of $\operatorname{sing}(f^{-1})$, which are not singularities of the inverse function. We say that $f$ has *bounded type* if $S(f)$ is bounded. Similarly, we say that $f$ has *finite type* if $S(f)$ is finite.

The next result relates the singular set and the postsingular set of a transcendental self-map $f$ of $\mathbb{C}^*$ with the corresponding sets of a lift $\tilde{f}$ of $f$, which is a transcendental entire function satisfying $\exp \circ \tilde{f} = f \circ \exp$. The proof is straightforward and we omit it.

**Lemma 2.1.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ and let $\tilde{f}$ be a lift of $f$. Then $S(\tilde{f}) = \exp^{-1}(S(f))$ and $P(\tilde{f}) \subseteq \exp^{-1}(P(f))$.*

Recall that if $f$ is a holomorphic self-map of $\mathbb{C}^*$, we define $\operatorname{ind}(f)$ to be the index of $f(\gamma)$ with respect to the origin, where $\gamma$ is any positively oriented simple closed curve around the origin. Observe that, in the hypothesis of the previous lemma, if $|\operatorname{ind}(f)|=1$, then $P(\tilde{f})=\exp^{-1}(P(f))$.

The following lemma is a basic property about the singular values of the composition of two functions.

**Lemma 2.2.** *Let $f$ and $g$ be meromorphic functions on $\mathbb{C}$. Then we have that $\operatorname{CP}(g\circ f)=\operatorname{CP}(f)\cup f^{-1}(\operatorname{CP}(g))$, $\operatorname{CV}(g\circ f)\subseteq g(\operatorname{CV}(f))\cup \operatorname{CV}(g)$ and $\operatorname{AV}(g\circ f)=g(\operatorname{AV}(f))\cup \operatorname{AV}(g)$.*

*Proof.* By the chain rule, $(g\circ f)'(z)=g'(f(z))f'(z)$, and thus

$$\begin{aligned}\operatorname{CP}(g\circ f)&=\operatorname{CP}(f)\cup f^{-1}(\operatorname{CP}(g)),\\ \operatorname{CV}(g\circ f)&=(g\circ f)(\operatorname{CP}(g\circ f))\\ &\subseteq (g\circ f)\operatorname{CP}(f)\cup g(\operatorname{CP}(g))\\ &=g(\operatorname{CV}(f))\cup\operatorname{CV}(g).\end{aligned}$$

Observe that the set $f^{-1}(\operatorname{CP}(g))$ may be empty, and hence the other inclusion does not hold in general.

Finally, if $\gamma$ is an asymptotic path of $g\circ f$ with asymptotic value $a$, then either $f(\gamma(t))\to b\in\operatorname{AV}(f)$ as $t\to+\infty$, where $g(b)=a$, or $f(\gamma)$ is an asymptotic path of $g$ and $a\in\operatorname{AV}(g)$. Therefore $\operatorname{AV}(g\circ f)\subseteq g(\operatorname{AV}(f))\cup\operatorname{AV}(g)$ and the opposite inclusion follows easily. □

Let $\mathcal{B}$ and $\mathcal{B}^*$ be the classes of bounded-type functions defined in the introduction. Observe that, by Lemma 2.2, both $\mathcal{B}$ and $\mathcal{B}^*$ are closed under composition. Recall that Theorem 1.2 establishes a way to construct functions in $\mathcal{B}^*$ from functions in $\mathcal{B}$. To prove this theorem, we need the following preliminary result.

**Proposition 2.3.** *Let $f(z)=z^n\exp\big(g(z)+h(1/z)\big)$ with $n\in\mathbb{Z}$ and $g,h$ nonconstant entire functions. If $f_\infty(z):=z^n\exp\big(g(z)\big)$ and $f_0(z):=z^n\exp\big(-h(z)\big)$ as well as $1/f_\infty$ and $1/f_0$ have bounded type, then $f\in\mathcal{B}^*$.*

Note that if $n\geqslant 0$, then $f_\infty$ and $f_0$ are transcendental entire functions, while if $n<0$, then they are meromorphic functions on $\mathbb{C}$ with a pole at the origin (which is omitted).

*Proof of Proposition 2.3.* We can express

$$f(z)=z^n\exp\big(g(z)+h(1/z)\big)=f_\infty(z)\cdot\exp\big(h(1/z)\big).$$

Suppose that $f_\infty(z)$ tends to a finite value $a\in\mathbb{C}$ as $z\to\infty$ along an asymptotic path $\gamma$. Then $f(z)\to e^{h(0)}a$ as $z\to\infty$ along $\gamma$. Conversely, if $f(z)$ tends to a finite value $a\in\mathbb{C}$ as $z\to\infty$ along an asymptotic path $\gamma$, then $f(z)\to a/e^{h(0)}$ as $z\to\infty$ along $\gamma$. Hence we have $\operatorname{AV}_\infty(f)=e^{h(0)}\cdot\operatorname{AV}(f_\infty)$.

Differentiating $f$, we obtain

$$f'(z)=f(z)\left(-\frac{h'(1/z)}{z^2}+\frac{f_\infty'(z)}{f_\infty(z)}\right),$$

or, equivalently,

$$\frac{zf'(z)}{f(z)}=-\frac{h'(1/z)}{z}+\frac{zf_\infty'(z)}{f_\infty(z)}.$$

It follows easily from [17, Lemma 1] that if $f \in \mathcal{B}$ then there is a constant $R_0 > 0$ such that

$$\left| z\frac{f'(z)}{f(z)} \right| \geqslant \frac{1}{4\pi}\big(\log|f(z)| - \log R_0\big), \qquad \text{for } z \in D(0, R_0), \tag{3}$$

and hence

$$\eta_f := \lim_{R\to+\infty} \inf \left\{ \left| z\frac{f'(z)}{f(z)} \right| \ : \ |f(z)| > R \right\} = +\infty. \tag{4}$$

If $n < 0$, the function $f_\infty$ is meromorphic but, since the pole at $z = 0$ is omitted and $\operatorname{sing}(f_\infty^{-1})$ is bounded away from the origin, the same proof of Lemma 3.6 can be used to obtain inequality (3) in this case as well. Suppose that $f_\infty$ has bounded type, then

$$\inf \left\{ \left| z\frac{f'_\infty(z)}{f_\infty(z)} \right| \ : \ |f_\infty(z)| > R \right\} \to +\infty \qquad \text{as } R \to +\infty.$$

Since for $n \geqslant 0$, $f_\infty$ is entire, the components of the set $\{z \in \mathbb{C} \ : \ |f_\infty(z)| > R\}$ are all unbounded and tend to infinity as $R \to +\infty$ in the sense that their distance from the origin tends to infinity. Therefore, since

$$\exp(h(1/z)) \to \exp(h(0)) \quad \text{ and } \quad \frac{h'(1/z)}{z} \to 0 \quad \text{ as } z \to \infty,$$

there exists $M, N > 0$ such that if $|f(z)| > R$ and $|z| \geqslant 1$, then

$$|f_\infty(z)| = \frac{|f(z)|}{\exp\big(\operatorname{Re} h(1/z)\big)} > \frac{R}{M} \quad \text{ and } \quad \left|\frac{h'(1/z)}{z}\right| < N,$$

and so

$$\inf \left\{ \left| z\frac{f'(z)}{f(z)} \right| : |f(z)|{>}R, \ |z|{\geqslant}1 \right\} \geqslant \inf \left\{ \left| z\frac{f'_\infty(z)}{f_\infty(z)} \right| : |f_\infty(z)|{>}\frac{R}{M} \right\} - N \to +\infty$$

as $R \to +\infty$. Hence, $\mathrm{CV}(f)$ cannot contain a sequence of critical values whose critical points are in $\mathbb{C} \setminus \mathbb{D}$ that accumulate to infinity, because if $f(z)$ is a critical value, then the quantity $zf'(z)/f(z) = 0$. Similarly, in a neighbourhood of zero,

$$\inf \left\{ \left| z\frac{f'(z)}{f(z)} \right| : |f(z)|{<}\frac{1}{R}, \ |z|{\leqslant}1 \right\} \geqslant \inf \left\{ \left| z\frac{f'_0(z)}{f_0(z)} \right| : |f_0(z)|{>}\frac{R}{M'} \right\} - N' \to +\infty$$

as $R \to +\infty$, and thus $f$ has no critical values accumulating to zero whose critical points are in $\mathbb{D}$. Finally, since we are assuming that the functions $1/f_\infty$ and $1/f_0$ have bounded type too, $0 \notin \operatorname{sing}(f_\infty^{-1})' \cup \operatorname{sing}(f_0^{-1})'$, so the sets

$$\inf \left\{ \left| z\frac{f'(z)}{f(z)} \right| : |f(z)|{<}\frac{1}{R}, \ |z|{\geqslant}1 \right\}, \ \inf \left\{ \left| z\frac{f'(z)}{f(z)} \right| : |f(z)|{>}R, \ |z|{\leqslant}1 \right\} \to +\infty$$

as $R \to +\infty$. Hence $f \in \mathcal{B}^*$. □

Sixsmith [47] showed that if $f \notin \mathcal{B}$, then $\eta_f = 0$, where $\eta_f$ is the quantity defined in (4), and thus provided and alternative characterisation of functions in the class $\mathcal{B}$. This was later generalised by Rempe-Gillen and Sixsmith in [41].

Theorem 1.2 states that if $g, h \in \mathcal{B}$, then the function $f(z) = \exp\big(g(z) + h(1/z)\big)$ is in the class $\mathcal{B}^*$. Thus, it can be used to produce examples of functions in the class $\mathcal{B}^*$ from functions in the class $\mathcal{B}$ (see Example 2.6). Recall that Keen proved that if $g$ and $h$ are polynomials and $n \in \mathbb{Z}$, then $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$ is in the class $\mathcal{B}^*$ as well (see Proposition 4.5 and Lemma 4.6).

*Proof of Theorem 1.2.* Let $f_\infty = \exp\circ g$ where $g \in \mathcal{B}$. By Lemma 2.2,

$$\begin{aligned}\mathrm{AV}(f_\infty) &= \mathrm{AV}(\exp) \cup \exp(\mathrm{AV}(g)) = \exp(\mathrm{AV}(g)) \cup \{0\},\\ \mathrm{CP}(f_\infty) &= \mathrm{CP}(g) \cup g^{-1}(\mathrm{CP}(\exp)) = \mathrm{CP}(g) \cup g^{-1}(\emptyset) = \mathrm{CP}(g),\end{aligned}$$

and both $\mathrm{CV}(f_\infty) = \exp(\mathrm{CV}(g))$ and $\mathrm{AV}(f_\infty)$ are bounded in $\mathbb{C}$. On the other hand,

$$\begin{aligned}\mathrm{AV}(1/f_\infty) &= \mathrm{AV}(\exp) \cup \exp(\mathrm{AV}(-g)) = \exp(-\mathrm{AV}(g)) \cup \{0\},\\ \mathrm{CP}(1/f_\infty) &= \mathrm{CP}(-g) = \mathrm{CP}(g),\end{aligned}$$

and therefore $\mathrm{CV}(1/f_\infty) = \exp\big(-\mathrm{CV}(g)\big)$ and $\mathrm{AV}(1/f_\infty)$ are bounded in $\mathbb{C}$ too. Similarly, since $h \in \mathcal{B}$ the functions $f_0(z) = \exp\big(-h(z)\big)$ and $1/f_0$ have bounded type. Therefore $f_\infty$ and $f_0$ satisfy the hypothesis of Proposition 2.3 and the function $f(z) = \exp\big(g(z) + h(1/z)\big)$ is in the class $\mathcal{B}^*$. □

**Remark 2.4.** Observe that if $n \neq 0$ and $f(z) = z^n \exp\big(g(z)\big)$ with $g \in \mathcal{B}$, even if $\mathrm{CV}(g)$ is bounded the set $\mathrm{CV}(f)$ may accumulate to zero ($n > 0$) or to infinity ($n < 0$). Thus Theorem 1.2 is optimal.

**Remark 2.5.** The converse of Theorem 1.2 is not true in general as the critical values of $g$ can be unbounded in a vertical band and the critical values of $f_\infty$ be bounded in an annulus. For example, the Fatou function $g(z) = z + 1 + e^{-z}$ is not in the class $\mathcal{B}$, while the function $f(z) = \exp\big(g(z) + 1/z\big)$ is in the class $\mathcal{B}^*$ by Proposition 2.3 as $\mathrm{CV}(e^g) = \{e^2\}$ and $\mathrm{AV}(e^g) = \{0\}$.

**Example 2.6.** We give a couple of examples of functions in the class $\mathcal{B}^*$ constructed from functions in the class $\mathcal{B}$ using Theorem 1.2.
(i) The function $f(z) = \exp\big((\sin z + 1)/z\big)$ is in the class $\mathcal{B}^*$ and the set $\mathrm{sing}(f^{-1})$ contains infinitely many points that accumulate at $z = 1$.
(ii) The function $f(z) = \exp(\exp z + 1/z)$ is in the class $\mathcal{B}^*$ and has a finite asymptotic value $a = 1$.

3. **Logarithmic coordinates for the class $\mathcal{B}^*$.** Let $f$ be a transcendental entire function or a transcendental self-map of $\mathbb{C}^*$. Let $a \in \hat{\mathbb{C}}$ and let $\hat{D}(a, r)$ denote the disc centred at $a$ of radius $r > 0$ in the spherical metric. For $r > 0$, choose $U(r)$ to be a connected component of $f^{-1}(D(a, r))$ such that if $0 < r_1 < r_2$, then $U(r_1) \subseteq U(r_2)$. We say that $U$ is a *logarithmic singularity* over $a$ if

$$f : U(r) \to D(a, r) \setminus \{a\}$$

is a universal covering for some $r > 0$ (see [23] for a classification of the singularities of the inverse). Transcendental self-maps of $\mathbb{C}^*$ have logarithmic singularities over both zero and infinity.

**Definition 3.1** (Logarithmic tract)**.** Let $f \in \mathcal{B}^*$ and let $A \subseteq \mathbb{C}$ be a topological annulus bounded away from zero and infinity that contains the set $S(f)$. Denote $W = \mathbb{C}^* \setminus A = W_0 \sqcup W_\infty$, where $W_0$ and $W_\infty$ are the components of $\mathbb{C}^* \setminus A$ whose closure in $\hat{\mathbb{C}}$ contains, respectively, zero and infinity. A *(logarithmic) tract* of $f$ is a connected component of $\mathcal{V} = f^{-1}(W)$.

Note that if $V$ is a tract of $f$, then the map $f : V \to W_i$ is a universal covering, where $i \in \{0, \infty\}$. The following lemma is a well-known classification of the coverings of the punctured disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ [20, Theorem 5.10]. If $X$ is a Riemann surface, we say that two holomorphic coverings $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ of $X$ are *equivalent* if there exists a conformal map $p_{21} : \widetilde{X_2} \to \widetilde{X_1}$ such that $p_2 = p_1 \circ p_{21}$.

**Lemma 3.2** (Coverings of $\mathbb{D}^*$)**.** *Let $U \subseteq \hat{\mathbb{C}}$ and let $f : U \to \mathbb{D}^*$ be a holomorphic covering. Then either $U$ is conformally equivalent to $\mathbb{D}^*$ and $f$ is equivalent to $z^d$, or $U$ is simply connected and $f$ is a universal covering and hence equivalent to the exponential map.*

In particular, the closure of each tract in $\hat{\mathbb{C}}$ contains only one of the essential singularities. Now we are going to introduce a logarithmic change of variables.

**Definition 3.3** (Logarithmic coordinates)**.** Let $f \in \mathcal{B}^*$ and consider $\mathcal{T} := \exp^{-1}(\mathcal{V})$ and $H := \exp^{-1}(W) = H_0 \sqcup H_\infty$ where $H_0 = \exp^{-1}(W_0)$ and $H_\infty = \exp^{-1}(W_\infty)$ contain, respectively, a left and a right half-plane. A *logarithmic transform* of $f$ is a continuous function $F : \mathcal{T} \to H$ which makes the following diagram commute (see Figure 2).

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[d, "\exp"'] & H \arrow[d, "\exp"] \\ \mathcal{V} \arrow[r, "f"'] & W \end{tikzcd}$$

The connected components of $\mathcal{T}$ are called *tracts* of $F$ and can be classified into four types
$$\mathcal{T} =: \mathcal{T}_0^0 \sqcup \mathcal{T}_0^\infty \sqcup \mathcal{T}_\infty^0 \sqcup \mathcal{T}_\infty^\infty,$$
where the lower index indicates if the tracts have zero or infinity in their closure and the upper index indicates if they are mapped to $H_0$ or $H_\infty$ by $F$. We define $\mathcal{T}_0 := \mathcal{T}_0^0 \sqcup \mathcal{T}_0^\infty$ and $\mathcal{T}_\infty := \mathcal{T}_\infty^0 \sqcup \mathcal{T}_\infty^\infty$.

In the entire case, often the expressions 'lift' and 'logarithmic transform' are used indistinctly to refer to $F$ defined on the tracts. In this paper we reserve the word *lift* for an entire function $\tilde{f}$ such that $\exp \circ \tilde{f} = f \circ \exp$.

**Remark 3.4.** Observe that we can obtain $F$ as the restriction of a lift $\tilde{f}$ of $f$ to the set $\mathcal{T}$. However, since $F$ is only defined on $\mathcal{T}$, we can add a different integer multiple of $2\pi i$ to $F$ on each tract $T$, and hence $F$ is not necessarily the restriction of a transcendental entire function $\tilde{f}$.

**Theorem 3.5.** *If $f \in \mathcal{B}^*$, then a logarithmic transform $F : \mathcal{T} \to H$ of $f$ satisfies the following properties:*

*(a) the set $H$ is the disjoint union of two $2\pi i$-periodic Jordan domains $H_0$ and $H_\infty$ containing, respectively, a left and a right half-plane;*
*(b) every tract of $F$ is an unbounded Jordan domain whose points have real part either bounded from below and unbounded from above or unbounded from below and bounded from above;*
*(c) the tracts of $F$ have disjoint closures and accumulate only at zero and infinity;*
*(d) for every tract $T$ of $F$, the function $F_{|T} : T \to H$ is a conformal isomorphism;*
*(e) for every tract $T$ of $F$, the function $\exp_{|T}$ is injective;*
*(f) the set $\mathcal{T}$ is invariant under translation by $2\pi i$.*

*Moreover, there exists a curve $\delta \subseteq \mathbb{C}^* \setminus \overline{\mathcal{V}}$ joining zero to infinity, where $\mathcal{V} = \exp \mathcal{T}$.*

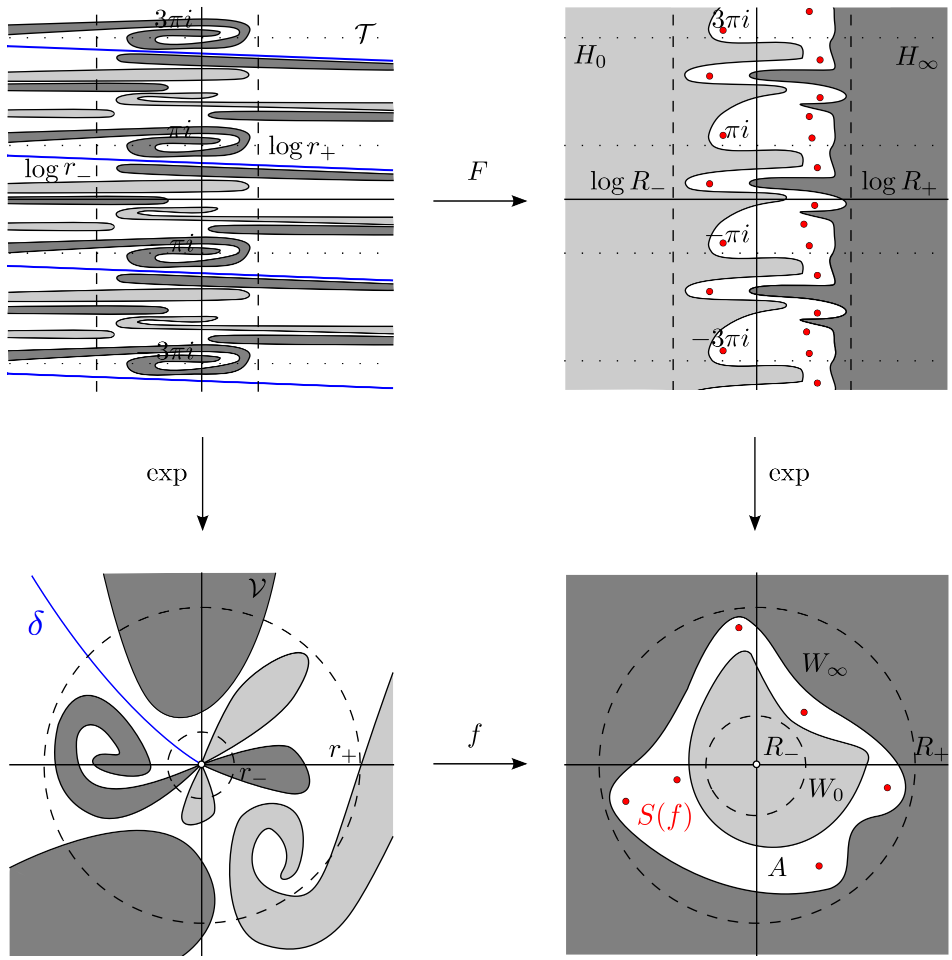


FIGURE 2. Logarithmic coordinates for a function $f \in \mathcal{B}^*$.

*Proof.* These properties follow easily from the fact that the exponential map is a holomorphic cover and, in particular, a local homeomorphism. The fact that there exists a curve $\delta \subseteq \mathbb{C}^* \setminus \overline{\mathcal{V}}$ joining zero to infinity is a straightforward consequence of (b) and (c) in the case that $\mathcal{V}$ consists of finitely many tracts. Otherwise, this follows from Carathéodory's theorem and the fact that $\mathcal{V}$ is locally connected (see [4, Lemma 2.1]). Hence, we can define a continuous branch of the logarithm on the set $\mathbb{C}^* \setminus \delta$ and, in particular, on $\overline{\mathcal{V}}$. □

We denote by $\mathcal{B}^*_{log}$ the class of holomorphic functions $F : \mathcal{T} \to H$ satisfying properties (a) to (f) in Theorem 3.5, regardless of whether they arise as a logarithmic transform of a function $f \in \mathcal{B}^*$ or not. The main advantage of working in the class $\mathcal{B}_{log}$ defined in [44] or, in our case, the class $\mathcal{B}^*_{log}$ is that functions satisfy the following expansivity property (5) which implies that points in $I(f)$ eventually escape at an exponential rate.

**Lemma 3.6.** *Let $F : \mathcal{T} \to H$ be a function in the class $\mathcal{B}^*_{log}$. There exists $R > 0$ sufficiently large such that if $|\operatorname{Re} F(z)| \geqslant R$, then*

$$|F'(z)| \geqslant \frac{1}{4\pi} |\operatorname{Re} F(z)| - R.$$

*In particular, there exists $R_0 = R_0(F) > 0$ so that*

$$|F'(z)| \geqslant 2 \quad \textit{for } |\operatorname{Re} F(z)| \geqslant R_0. \tag{5}$$

See [17, Lemma 1] for the original result for entire functions. The proof relies on properties (a), (d) and (e) of logarithmic transforms, which are common in both settings, and Koebe's 1/4-theorem.

Sullivan proved that rational maps have no wandering domain [48]. Following this result, Keen [24], Kotus [26] and Makienko [28] proved independently that transcendental self-maps of $\mathbb{C}^*$ with finitely many singular values have no wandering domains. In [26], Kotus also showed that finite-type maps in $\mathbb{C}^*$ have no Baker domains. Here we show that bounded-type functions have no escaping Fatou component adapting the proof that Eremenko and Lyubich gave for class $\mathcal{B}$ [17, Theorem 1].

*Proof of Theorem 1.1.* Suppose to the contrary that there exists $z_0 \in F(f) \cap I(f)$. Then, by normality, there is some $R_0 > 0$ so that $B_0 := B(z_0, R) \subseteq F(f) \cap I(f)$. Since $B_0 \subseteq I(f)$, there exists $N_0 \in \mathbb{N}_0$ such that the sets $B_n := f^n(B_0)$, $n \in \mathbb{N}$, are contained in the union $\mathcal{V}$ of the tracts of $f$ for all $n \geqslant N_0$; we can assume without loss of generality that $N_0 = 0$. Let $C_0$ be a connected component of $\exp^{-1}(B_0)$ and put $C_n := F^n(C_0)$ for $n \in \mathbb{N}$. For every $R > 0$, there exists $N = N(R) \in \mathbb{N}_0$ such that

$$C_n \subseteq \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| > R\} \quad \text{for all } n > N.$$

Take any point $\zeta_0 \in C_0$ and set $\zeta_n := F^n(\zeta_0) \in C_n$ and $d_n := \operatorname{dist}(\zeta_n, \partial C_n)$ for $n \in \mathbb{N}$. Then Koebe's 1/4-theorem implies that

$$d_{n+1} \geqslant \frac{1}{4} d_n |F'(\zeta_n)| \quad \text{for all } n \in \mathbb{N}.$$

Since $|\operatorname{Re} F(\zeta_n)| \to +\infty$ as $n \to \infty$, by Lemma 3.6, we have $|F'(\zeta_n)| \to +\infty$ and hence $d_n \to +\infty$. But this contradicts property (e) of functions in the class $\mathcal{B}^*_{log}$ because $\mathcal{T}$ does not contain any vertical segment of length $2\pi$. Thus $F(f) \cap I(f) = \emptyset$ and $I(f) \subseteq J(f)$. $\square$

By property (a) in Theorem 3.5, if $F : \mathcal{T} \to H$ is in the class $\mathcal{B}^*_{log}$, then the set $H$ contains the union of two half-planes of the form

$$\mathbb{H}^{\pm}_R := \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| > R\} = \mathbb{H}^-_R \cup \mathbb{H}^+_R$$

for some $R > 0$. We call the function $F$ *normalised* if the tracts of $F$ do not intersect the imaginary axis, $H = \mathbb{H}^{\pm}_R$ for some $R > 0$ and $F$ satisfies the expansivity property (5).

**Definition 3.7** (Normalisation)**.** We say that a logarithmic transform $F : \mathcal{T} \to H$ in $\mathcal{B}^*_{log}$ is *normalised* if $\overline{\mathcal{T}} \cap \{z \in \mathbb{C} \ : \ \operatorname{Re} z = 0\} = \emptyset$, the set $H = \mathbb{H}^{\pm}_R$ for some $R > 0$ and the expansivity property (5) is satisfied in all $H$. We denote this class of functions by $\mathcal{B}^{*n}_{log}$.

Logarithmic transforms of transcendental entire functions can be normalised so that $H$ is the right half-plane $\mathbb{H}$. In contrast, in the punctured plane, when we say that $F$ is normalised we need to specify the constant $R$. The next lemma shows that we can always assume that $F$ is in the class $\mathcal{B}_{log}^{*n}$ by restricting the function to a smaller set.

**Lemma 3.8.** *Let $F:\mathcal{T}\to H$ be a function in the class $\mathcal{B}_{log}^{*}$. There exists a constant $R=R(F)>0$ such that $\mathbb{H}_R^{\pm}\subseteq H$ and the restriction of $F$ to $F^{-1}(\mathbb{H}_R^{\pm})$ is a normalised logarithmic transform.*

*Proof.* Suppose that $F$ is not normalised. Let $\{B_n\}_{n\in\mathbb{Z}}$, denote the connected components of the set $\mathbb{C}\setminus\exp^{-1}(\delta)$, where $\delta$ is the curve from Theorem 3.5. For $n\in\mathbb{N}$, the sets

$$X_n=\mathcal{T}_0\cap B_n\cap\mathbb{H}^+ \quad \text{and} \quad Y_n=\mathcal{T}_\infty\cap B_n\cap\mathbb{H}^-$$

are bounded and hence their images $F(X_n)$ and $F(Y_n)$ have bounded real part. All the sets $F(X_n)$ and $F(Y_n)$, $n\in\mathbb{N}$, are vertical translates of $F(X_0)$ and $F(Y_0)$ and hence $F(\mathcal{T}_0\cap\mathbb{H}^+)$ and $F(\mathcal{T}_\infty\cap\mathbb{H}^-)$ have bounded real part. Therefore, there exists $R_1>0$ sufficiently large such that

$$\big(F(\mathcal{T}_0\cap\mathbb{H}^+)\cup F(\mathcal{T}_\infty\cap\mathbb{H}^-)\big)\cap\mathbb{H}_{R_1}^{\pm}=\emptyset.$$

Then, if $R_0=R_0(F)>0$ is the constant from Lemma 3.6 so that $|F'(z)|>2$ if $|\operatorname{Re}F(z)|\geqslant R_0$, it is enough to put $R:=\max\{R_0,R_1\}$. □

The following lemma is a stronger version of the expansivity property (5) for functions in $\mathcal{B}_{log}^{*n}$, and says that escaping orbits eventually separate at an exponential rate. The construction in the proof of [44, Lemma 3.1] can be adapted easily to this setting.

**Lemma 3.9.** *Let $F:\mathcal{T}\to H$ be a function in the class $\mathcal{B}_{log}^{*n}$ with $H=\mathbb{H}_R^{\pm}$ for some $R>0$. If $T$ is a tract of $F$ and $z,w\in T$ are such that $|z-w|\geqslant 8\pi$, then*

$$|F(z)-F(w)|\geqslant\exp\left(\frac{|z-w|}{8\pi}\right)\cdot\min\{|\operatorname{Re}F(z)|-R,\ |\operatorname{Re}F(w)|-R\}.$$

Next we introduce a subclass of $\mathcal{B}_{log}^{*}$ consisting of the functions $F:\mathcal{T}\to H$ for which the image $F(\mathcal{T})$ covers the whole $\overline{\mathcal{T}}$, which have nicer properties.

**Definition 3.10** (Disjoint type)**.** We say that a function $F:\mathcal{T}\to H$ in the class $\mathcal{B}_{log}^{*}$ is of *disjoint type* if $\overline{\mathcal{T}}\subseteq H$.

If $f\in\mathcal{B}^*$ and $A=\mathbb{C}^*\setminus W$ is an annulus containing $S(f)$, then $f(\mathbb{C}^*\setminus\mathcal{T})\subseteq A$, where $\mathcal{T}=f^{-1}(W)$. In the case that $f$ has a logarithmic transform $F$ that is of disjoint type (with $H=\exp^{-1}(W)$), we have $A\subseteq\mathbb{C}^*\setminus\mathcal{V}$ and $f(A)\subseteq A$. Hence $A\subseteq F(f)$ and it follows from the classification of Fatou components that, in this situation, $F(f)$ consists of a single doubly connected component $U$ which is the immediate basin of attraction of an attracting fixed point in $A$.

**Remark 3.11.** Independently of [44], Barański showed that the Julia set of disjoint-type maps in the class $\mathcal{B}$ consists of disjoint hairs that are homeomorphic to $[0,+\infty)$ (we call them dynamic rays) and that the endpoints of these hairs are the only points in $J(f)$ accessible from $F(f)$ [2, Theorem C].

**Example 3.12.** The function $f(z)=\exp\big(0.3(z+1/z)\big)$ is in the class $\mathcal{B}^*$ and has a logarithmic transform of disjoint type (see Figure 3).

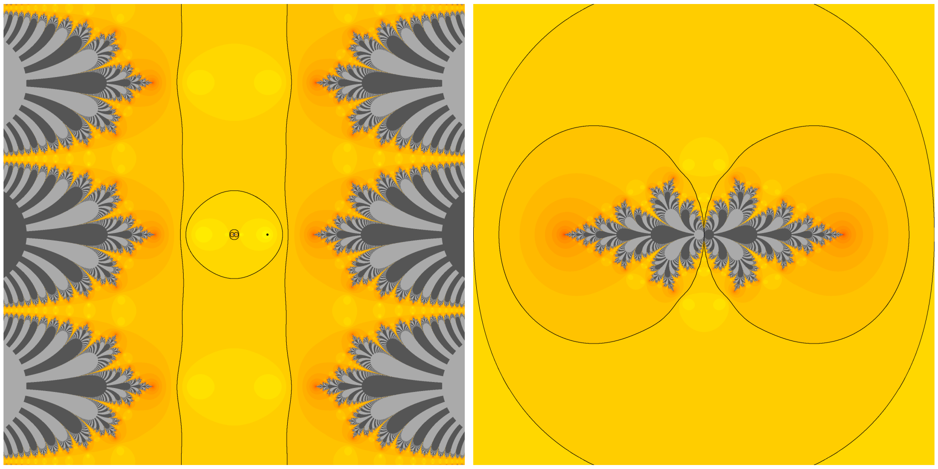

FIGURE 3. Phase space of the function $f(z) = \exp(0.3(z + 1/z))$ which has a disjoint-type logarithmic transform (see Example 3.12). In orange, the basin of attraction of the fixed point $z_0 \simeq 2.2373$. Left, $z \in [-16, 16] + i[-16, 16]$; right, $z \in [-0.3, 0.3] + i[-0.3, 0.3]$.

Sometimes tracts exhibit better geometric properties that make them easier to study. In the next section we will see that this is the case for transcendental self-maps of $\mathbb{C}^*$ of finite order.

**Definition 3.13** (Good geometry properties)**.** Let $F \in \mathcal{B}^*_{log}$ and let $T$ be a tract of $F$.

(a) We say that $T$ has *bounded wiggling* if there exist $K > 1$ and $\mu > 0$ such that for every $z_0 \in \overline{T}$, every point $z$ on the hyperbolic geodesic of $T$ that connects $z_0$ to infinity satisfies

$$|\operatorname{Re} z| > \frac{1}{K}|\operatorname{Re} z_0| - \mu.$$

In the case $K = 1$ and $\mu = 0$ we say that $T$ has *no wiggling*. A function $F \in \mathcal{B}^*_{log}$ has *uniformly bounded wiggling* if the wiggling of all tracts of $F$ is bounded by the same constants $K, \mu$.

(b) We say that $T$ has *bounded slope* if there exist constants $\alpha, \beta > 0$ such that

$$|\operatorname{Im} z - \operatorname{Im} w| \leqslant \alpha \max\{|\operatorname{Re} z|, |\operatorname{Re} w|\} + \beta$$

for all $z, w \in T$. Equivalently, $T$ contains a curve $\gamma : [0, \infty) \to T$ such that $|F(\gamma(t))| \to +\infty$ and

$$\limsup_{t\to\infty} \frac{|\operatorname{Im} \gamma(t)|}{|\operatorname{Re} \gamma(t)|} < \infty.$$

We say that $T$ has *zero slope* if this limit is zero.

We say $F$ has *good geometry* if the tracts of $F$ have bounded slope and uniformly bounded wiggling.

**Remark 3.14.** (i) Observe that it is enough that a tract $T$ from $\mathcal{T}_\alpha$, $\alpha \in \{0, \infty\}$, has bounded slope to ensure that all tracts in $\mathcal{T}_\alpha$ do. We can use the same constants $(\alpha, \beta)$ for $\mathcal{T}_\infty$ and $\mathcal{T}_0$: if they have bounded slope with different values $(\alpha_1,\beta_1)$ and $(\alpha_2,\beta_2)$, then it is enough to take $\alpha := \max\{\alpha_1,\alpha_2\}$ and $\beta := \max\{\beta_1,\beta_2\}$.

(ii) If $F, G \in \mathcal{B}^{*n}_{log}$ and $G$ has bounded slope, then $G\circ F$ has bounded slope with the same constants as $G$.

4. **Order of growth in $\mathbb{C}^*$.** Recall that the *order* of an entire function is defined to be the infimum of $\rho \in \mathbb{R}\cup\{\infty\}$ such that $\log|f(z)| = \mathcal{O}(|z|^\rho)$ as $z\to\infty$. Equivalently,

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$

where

$$M(r,f) := \max_{|z|=r} |f(z)| < +\infty.$$

Polynomials have order zero and the function $\exp(z^k)$, $k\in\mathbb{N}$, has order $k$. There are also transcendental entire functions of order zero and of infinite order.

When we deal with holomorphic self-maps of $\mathbb{C}^*$, controlling the growth means looking at how $|f(z)|$ tends to zero or infinity when $z$ approaches zero or infinity. Observe that if $f$ is such map, then $1/f$ is also holomorphic on $\mathbb{C}^*$, and since $f$ has no zeros

$$m(r,f) := \min_{|z|=r} |f(z)| = \frac{1}{M(r,1/f)} > 0.$$

As before, for simplicity, we will write $M(r)$ and $m(r)$ when it is clear what the function $f$ is.

A priori, the notion of order of growth in this context involves the following four quantities:

$$\rho^\infty_{\max}(f) := \limsup_{r\to+\infty} \frac{\log\log M(r)}{\log r}, \qquad \rho^\infty_{\min}(f) := \limsup_{r\to+\infty} \frac{\log\big(-\log m(r)\big)}{\log r},$$

$$\rho^0_{\max}(f) := \limsup_{r\to 0} \frac{\log\log M(r)}{-\log r}, \qquad \rho^0_{\min}(f) := \limsup_{r\to 0} \frac{\log\big(-\log m(r)\big)}{-\log r}.$$

However, if an entire function $f$ has no zeros, then $\rho(f) = \rho(1/f)$ as a consequence of the fact that you can write the order in terms of the Nevanlinna characteristic function $T(R,f)$:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and Jensen's formula says that

$$T(r,f) = T(r,1/f) + \log|f(0)|$$

(see section 1.2 of [21]). It follows from the general expression of a transcendental self-map of $\mathbb{C}^*$

$$f(z) = z^n \exp(g(z) + h(1/z))$$

with $n\in\mathbb{Z}$ and $g,h$ non-constant entire functions that

$$\log|f(z)| = n\log|z| + \operatorname{Re} g(z) + \operatorname{Re} h(0) + o(1) \quad \text{as } z\to\infty,$$

and therefore

$$\log M(r,f) = \log M(r,e^g) + O(\log r) \quad \text{as } z\to\infty. \tag{6}$$

Note that in a neighbourhood of infinity the term $h(1/z)$ is not relevant and the same happens with $g(z)$ in a neighbourhood of the origin. Then, putting (6) into the four order quantities defined above and using Jensen's formula we obtain

$$\rho^{\infty}_{\max}(f) = \rho^{\infty}_{\max}(e^g) = \rho(e^g) = \rho^{\infty}_{\min}(e^g) = \rho^{\infty}_{\min}(f) \tag{7}$$

and, similarly, at zero

$$\rho^{0}_{\max}(f) = \rho^{\infty}_{\max}(e^h) = \rho(e^h) = \rho^{\infty}_{\min}(e^h) = \rho^{0}_{\min}(f),$$

so, in fact, the order of growth of $f$ involves only two quantities.

**Definition 4.1** (Order of growth)**.** Let $f$ be a transcendental self-map of $\mathbb{C}^*$ of the form

$$f(z) = z^n \exp\bigl(g(z) + h(1/z)\bigr)$$

with $n \in \mathbb{Z}$ and $g, h$ non-constant entire functions. We say that $f$ has *finite order* if both quantities

$$\rho_\infty(f) := \rho(e^g) \quad \text{and} \quad \rho_0(f) := \rho(e^h)$$

are finite.

**Example 4.2.** The functions $f(z) = z^n \exp(P(z) + Q(1/z))$, with $n \in \mathbb{Z}$ and $P, Q$ polynomials, are transcendental self-maps of $\mathbb{C}^*$ of finite order and $\rho_\infty(f) = \deg P$ and $\rho_0(f) = \deg Q$.

**Remark 4.3.** Keen [24] defined the order of transcendental self-maps of $\mathbb{C}^*$ using

$$\widetilde{M}(r, f) = \max_{z \in \partial A_r} |f(z)| \quad \text{and} \quad \widetilde{m}(r, f) = \min_{z \in \partial A_r} |f(z)|$$

for $r > 0$, where $A_r := \{z \in \mathbb{C} \;:\; 1/r < |z| < r\}$. It follows from the maximum principle that $\widetilde{M}(r, f)$ and $\widetilde{m}(r, f)$ are, respectively, the maximum and minimum of $|f(z)|$ in the whole annulus $A_r$ (in the same way that, for an entire function, we have $M(r) = \max_{z \in D(0,r)} |f(z)|$). In our notation,

$$\widetilde{M}(r, f) = \max\{M(r),\ M(1/r)\} \quad \text{and} \quad \widetilde{m}(r, f) = \min\{m(r),\ m(1/r)\}.$$

Now we will see that, in fact, every holomorphic self-map of $\mathbb{C}^*$ that has finite order necessarily has to be of the form given in Example 4.2. We will begin by stating a classical result concerning entire functions of finite order due to Pólya [35] (see also [21, Theorem 2.9]).

**Lemma 4.4.** *If $f$ is a non-constant entire function of finite order with no zeros, then $f(z) = \exp(h(z))$ and $h$ is a polynomial.*

Using Lemma 4.4, we obtain the following.

**Proposition 4.5.** *Every transcendental self-map of $\mathbb{C}^*$ of finite order is of the form*

$$f(z) = z^n \exp(P(z) + Q(1/z))$$

*for some $n \in \mathbb{Z}$ and $P, Q$ polynomials.*

Keen proved the stronger result that every topological conjugacy class of analytic self-maps of $\mathbb{C}^*$ contains a function of this form [25, Theorem 1], but we give a direct proof of Proposition 4.5 for completeness.

*Proof.* We know that every transcendental self-map of $\mathbb{C}^*$ is of the form

$$f(z) = z^n \exp\big(g(z) + h(1/z)\big)$$

for some $n \in \mathbb{Z}$ and $g, h$ non-constant entire functions. Thus, by (7),

$$\rho(e^g) = \rho_\infty(f) < +\infty$$

and so it follows from Lemma 4.4 that $g$ has to be a polynomial. On the other hand,

$$\rho(e^h) = \rho_0(f) < +\infty$$

and so $h$ has to be a polynomial as well. $\square$

Keen also showed that, in $\mathbb{C}^*$, finite order implies finite type [25, Proposition 2]. This is very different to what happens in the entire case, where we have functions of finite order in the class $\mathcal{B}$ that are not in the Speiser class $\mathcal{S}$ of finite-type transcendental entire functions. An example of such a function is given by $\sin(z)/z$ which has order one and infinitely many critical values in any open interval in $\mathbb{R}$ containing the origin. We state Keen's result for future reference.

**Lemma 4.6.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$. If $f$ has finite order with $\rho_\infty(f) = p$ and $\rho_0(f) = q$, then sing$(f^{-1})$ consists of at most $p + q$ critical values and the asymptotic values zero and infinity.*

Finally, we show that the tracts of finite order functions have a fairly simple geometry.

**Proposition 4.7.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ of finite order and let $F \in \mathcal{B}^{*n}_{log}$ be a logarithmic transform of $f$. Then $f$ has a finite number of tracts and the tracts of $F$ have zero slope and can be chosen to have no wiggling.*

*Proof.* Suppose that $\rho_\infty(f) = p$ and $\rho_0(f) = q$ with $p, q \geqslant 1$. Then, by Proposition 4.5,

$$f(z) = z^n \exp\big(P(z) + Q(1/z)\big),$$

where $n \in \mathbb{Z}$ and $P, Q$ are, respectively, polynomials of degree $p, q$. We focus on the tracts whose closure in $\hat{\mathbb{C}}$ contains infinity; the case where the closure contains zero is similar. We have

$$|f(z)| = \exp\big(\mathrm{Re}\,(az^p) + o(\mathrm{Re}\,(z^p))\big) \quad \text{as } z \to \infty, \tag{8}$$

where $a \in \mathbb{C}$. Let $\phi = \arg(a)$. The image of the imaginary axis under the exponential function is the unit circle, so it does not intersect the tracts of $f$ defined by $|f(z)| > R$ for large values of $R$. Thus, such tracts are contained in the sectors determined by the preimages of the imaginary axis by the map $az^p$, that is the radial lines of angle $(k\pi + \pi/2 - \phi)/p$, $k \in \mathbb{Z}$. Tracts that map to a neighbourhood of infinity lie in the sectors containing the radial lines of angle $(2k\pi - \phi)/p$, $0 \leqslant k < p$, while tracts that map to a neighbourhood of zero lie in the sectors containing the radial lines of angle $((2k+1)\pi - \phi)/p$, $0 \leqslant k < p$. The preimages of radial lines by the exponential function are horizontal lines and hence the tracts of $F$ are contained in horizontal bands and have zero slope.

Finally, by (8), the boundaries of the tracts tend asymptotically to the horizontal lines mentioned above and hence the tracts of $F$ can be chosen to have no wiggling if $R$ is sufficiently large. $\square$

It follows from Proposition 4.5 that, in the punctured plane, functions of finite order (as well as entire functions with no zeros) can only have integer orders $\rho_0(f)$ and $\rho_\infty(f)$. Recall that for every asymptotic value $a$ there is a curve to zero or infinity, called an asymptotic path of $f$, along which $f$ tends to $a$ (see Section 2). There are always exactly $2\rho_\infty(f)$ asymptotic paths to infinity corresponding, asymptotically, to the preimages of the positive (asymptotic value infinity) or negative (asymptotic value zero) real line by $az^d$ where $a \in \mathbb{C}$ and $d = \rho_\infty(f)$. Therefore the asymptotic paths alternate as you go around a circle of large radius (see Figure 4). Similarly, in a neighbourhood of zero there are $2\rho_0(f)$ asymptotic paths with the same structure. Each of these asymptotic paths is contained in a logarithmic tract and vice versa.

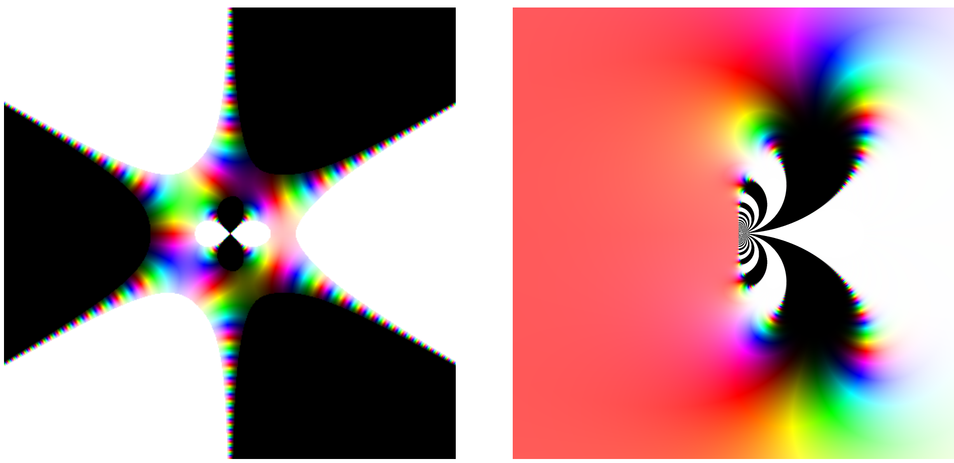

FIGURE 4. Logarithmic tracts of functions of finite order with $\rho_\infty(f) = 3$ and $\rho_0(f) = 2$ (left) and infinite order (right). The color of every point $z \in \mathbb{C}^*$ has been chosen according to the modulus (luminosity) and argument (hue) of $f(z)$.

Another basic property of entire functions in the class $\mathcal{B}$ is that they have lower order greater or equal than 1/2 [22] (see also [42, Lemma 3.5]). This is due to the fact that $f$ is bounded on a path $\delta$ to infinity. Note that $\delta$ can be chosen to be any path that lies in the complement of the tracts of $f$. Recall that the *lower order* of an entire function is

$$\lambda(f) := \liminf_{r \to +\infty} \frac{\log \log M(r, f)}{\log r}.$$

If $f$ is a transcendental self-map of $\mathbb{C}^*$ we consider

$$\lambda_\infty(f) := \liminf_{r \to +\infty} \frac{\log \log M(r, f)}{\log r} \quad \text{and} \quad \lambda_0(f) := \liminf_{r \to 0} \frac{\log \log 1/m(r, f)}{\log 1/r}.$$

Recall that Theorem 1.3 in the introduction states that, in this setting, we have $\lambda_0(f) = \rho_0(f)$ and $\lambda_\infty(f) = \rho_\infty(f)$. To prove this, we shall use the Borel-Carathéodory theorem in the form given in [49, Theorem 8].

**Lemma 4.8** (Borel-Carathéodory theorem). *Let $f$ be a transcendental entire function and define, for $r > 0$,*

$$B(r, f) := \min_{|z|=r} \operatorname{Re} f(z), \quad A(r, f) := \max_{|z|=r} \operatorname{Re} f(z).$$

*Then, there is $r_0 = r_0(f) > 0$ and $C = C(f) > 0$ such that*

$$B(r) \leqslant M(r) < \frac{R}{R-r}\big(4A(R) + C\big)$$

*for all $R > r > r_0$.*

*Proof of Theorem 1.3.* Let $f(z) = z^n \exp(g(z) + h(1/z))$ with $n \in \mathbb{Z}$ and $g, h$ nonconstant entire functions. We treat separately the cases where the function $f$ has finite order and infinite order. For simplicity, we only consider $\rho_\infty(f)$ and $\lambda_\infty(f)$; the proof for $\rho_0(f)$ and $\lambda_0(f)$ is completely analogous.

Suppose that $\rho_\infty(f) = p < +\infty$. Then, by Proposition 4.5, $g$ is a polynomial and, by (8),

$$\lambda_\infty(f) = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r} = \liminf_{r\to+\infty} \frac{\log A(r,g)}{\log r}.$$

Since $ar^p$, $a > 0$, is an increasing function for $r > 0$, it is clear that $\lambda_\infty(f) = \rho_\infty(f)$.

Now suppose that $\rho_\infty(f) = +\infty$. We use Lemma 4.8 with $R = 2r$: there is $C > 0$ and $r_0 > 0$ such that

$$M(r,g) < 2\big(4A(2r,g) + C\big) \quad \text{for all } r > r_0.$$

Therefore, since $g$ is a transcendental entire function, we have

$$\lambda_\infty(f) = \liminf_{r\to+\infty} \frac{\log A(r,g)}{\log r} \geqslant \liminf_{r\to+\infty} \frac{\log M(r/2,g)}{\log r} = \lim_{r\to+\infty} \frac{\log M(r,g)}{\log r} = +\infty$$

as required. □

Observe that if $F \in \mathcal{B}^*_{log}$, then the tracts of $F$ in each of the sets $\mathcal{T}_0$ and $\mathcal{T}_\infty$ can be ordered with respect to the vertical position around infinity. Therefore it makes sense to speak about a tract being in between two other tracts. This ordering is known as the *lexicographic order* and we will come back to it later (see Definition 5.9).

## 5. **Symbolic dynamics and combinatorics.**

Maps in class $\mathcal{B}^*_{log}$ are defined on a set $\mathcal{T}$, which is a union of tracts, and, therefore, the orbits of some points in $\mathcal{T}$ are truncated if $F^k(z) \notin \mathcal{T}$ for some $k \in \mathbb{N}$. We denote by $J(F)$ the set of points that can be iterated infinitely many times by $F$.

**Definition 5.1** (Julia set of $F$)**.** Let $F : \mathcal{T} \to H$ be a map in class $\mathcal{B}^*_{log}$. We define the *Julia set* of $F$ to be

$$J(F) := \{z \in \overline{\mathcal{T}} \ :\ F^n(z) \text{ is defined and in } \overline{\mathcal{T}} \text{ for all } n \in \mathbb{N}_0\},$$

and, for $K > 0$, we put

$$J^K(F) := \{z \in \overline{\mathcal{T}} \ :\ |\operatorname{Re} F^n(z)| \geqslant K \text{ for all } n \in \mathbb{N}_0\}.$$

As we will show in the following lemma, the reason why $J(F)$ is called the Julia set of $F$ is that points of $J(F)$ project to points in $J(f)$ by the exponential map. However, note that in the case that $F \in \mathcal{B}^*_{log}$ is the logarithmic transform of a function $f \in \mathcal{B}^*$, there exists an entire function $\tilde{f}$ that is a lift of $f$ and then $J(F) \subseteq J(\tilde{f}) = \exp^{-1} J(f)$ by a result of Bergweiler [6].

**Lemma 5.2.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ and let $F \in \mathcal{B}^*_{log}$ be a logarithmic transform of $f$. If $F \in \mathcal{B}^{*n}_{log}$, then* $\exp J(F) \subseteq J(f)$ *and, if $F$ is of disjoint type, then* $\exp J(F) = J(f)$.

*Proof.* Suppose to the contrary that $z_0 \in \exp J(F) \cap F(f) \neq \emptyset$. Then proceeding as in the proof of Theorem 1.1 we get a contradiction between the expansivity of $F$ given by (5) and the fact that $\mathcal{T}$ does not contain vertical segments of length $2\pi$. Note that in the normalised case we are using the expansivity with respect to the Euclidean metric, that is, $|F'(z)| \geqslant 2$ for all $z \in \mathcal{T}$ (see Lemma 3.6), while in the disjoint-type case we use the expansivity with respect to the hyperbolic metric on $H$ because $\mathcal{T}$ is compactly contained in $H$.

If $F$ is of disjoint type, the inclusion $J(f) \subseteq \exp J(F)$ follows from the fact that $f(\mathbb{C}^* \setminus \mathcal{V}) \subseteq A$ and hence $F(f)$ consists of the immediate basin of attraction of an attracting fixed point in $\mathbb{C}^* \setminus \mathcal{V}$, and so

$$J(f) = \mathbb{C}^* \setminus \bigcup_{n\in\mathbb{N}} f^{-n}(\mathbb{C}^* \setminus \mathcal{V}) = \exp J(F)$$

as required. □

If $f$ is a transcendental self-map of $\mathbb{C}^*$, then the escaping set $I(f)$ consists of all points that accumulate to $\{0, \infty\}$. Essential itineraries describe the way points escape and were introduced in [30]. Let us recall the definition here.

**Definition 5.3** (Essential itinerary)**.** Let $f$ be a transcendental self-map of $\mathbb{C}^*$. We define the *essential itinerary* of a point $z \in I(f)$ to be the symbol sequence $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ such that

$$e_n := \begin{cases} 0, & \text{if } |f^n(z)| \leqslant 1, \\ \infty, & \text{if } |f^n(z)| > 1, \end{cases}$$

for all $n \in \mathbb{N}_0$.

For each $e \in \{0, \infty\}^{\mathbb{N}_0}$, we denote by $I_e^{0,0}(f)$ the set of escaping points whose essential itinerary is *exactly* $e$,

$$I_e^{0,0} := \{z \in I(f) \ : \ \forall n \in \mathbb{N}_0, \ |f^n(z)| > 1 \Leftrightarrow e_n = \infty\},$$

and, for $\ell, k \in \mathbb{N}_0$, we define

$$I_e^{\ell,k} := \{z \in I(f) \ : \ \forall n \in \mathbb{N}_0, \ |f^{n+\ell}(z)| > 1 \Leftrightarrow e_{n+k} = \infty\} = f^{-\ell}\big(I^{0,0}_{\sigma^k(e)}(f)\big),$$

where $\sigma$ denotes the Bernoulli shift map. Finally, we denote by $I_e(f)$ the set of all escaping points whose essential itinerary is, *eventually, a shift of* $e$,

$$I_e(f) := \{z \in I(f) \ : \ \exists \ell, k \in \mathbb{N}_0, \ \forall n \in \mathbb{N}_0, \ |f^{n+\ell}(z)| > 1 \Leftrightarrow e_{n+k} = \infty\},$$

or, equivalently,

$$I_e(f) := \bigcup_{\ell\in\mathbb{N}_0} \bigcup_{k\in\mathbb{N}_0} I_e^{\ell,k}(f) = \bigcup_{\ell\in\mathbb{N}_0} \bigcup_{k\in\mathbb{N}_0} f^{-\ell}\big(I^{0,0}_{\sigma^k(e)}(f)\big).$$

We say that two sequences $e_1, e_2 \in \{0, \infty\}^{\mathbb{N}_0}$ are *equivalent* essential itineraries if $\sigma^m(e_1) = \sigma^n(e_2)$ for some $m, n \in \mathbb{N}_0$. If $e_1$ and $e_2$ are *not* equivalent, then $I_{e_1}(f) \cap I_{e_2}(f) = \emptyset$.

We now introduce the escaping set for maps in the class $\mathcal{B}^*_{log}$, which is a subset of the Julia set of $F$.

**Definition 5.4** (Escaping set of $F$)**.** Let $F : \mathcal{T} \to H$ be a map in the class $\mathcal{B}^*_{log}$. We define the *escaping set* of $F$ to be

$$I(F) := \{z \in J(F) \ : \ \lim_{n\to\infty} |\operatorname{Re} F^n(z)| = +\infty\} = J(F) \cap \exp^{-1} I(f).$$

In terms of $F$, a point $z \in I(F)$ has *essential itinerary* $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$ if $\operatorname{Re} F^n(z) \leqslant 0$ if and only if $e_n = 0$ for all $n \in \mathbb{N}_0$.

Observe that $\exp I(F) \subseteq I(f)$ and, in fact, every point in $I(f)$ eventually enters $\exp I(F)$. As with $J(F)$, if $f$ is a transcendental self-map of $\mathbb{C}^*$ and $\tilde{f}$ is a lift of $f$, then $I(F) \subseteq I(\tilde{f})$ but, in general, these sets are different, as $\tilde{f}$ may have points that escape in the imaginary direction, which correspond to bounded orbits for $f$.

For every function $F \in \mathcal{B}^*_{log}$, we denote by $\mathcal{A}$ (respectively $\mathcal{A}^0_0, \mathcal{A}^\infty_0, \mathcal{A}^0_\infty, \mathcal{A}^\infty_\infty$) the *symbolic alphabet* consisting of all tracts in $\mathcal{T}$ (respectively $\mathcal{T}^0_0, \mathcal{T}^\infty_0, \mathcal{T}^0_\infty, \mathcal{T}^\infty_\infty$; see Definition 3.3). We associate a symbol sequence $(T_n) \in \mathcal{A}^{\mathbb{N}_0}$ to each point $z \in J(F)$ that describes to which tract the iterate $F^n(z)$ belongs for all $n \in \mathbb{N}_0$.

**Definition 5.5** (External address of $F$)**.** Let $F \in \mathcal{B}^*_{log}$ and let $z \in J(F)$. We define the *external address* of $z$, $\operatorname{addr}_F(z)$, to be the symbol sequence $\underline{s} = (T_n) \in \mathcal{A}^{\mathbb{N}_0}$ such that $F^n(z) \in \overline{T_n}$ for all $n \in \mathbb{N}_0$.

**Remark 5.6.** Let $F$ be a normalised logarithmic transform. Then the Bernoulli shift map $\sigma : \mathcal{A}^{\mathbb{N}_0} \to \mathcal{A}^{\mathbb{N}_0}$ mapping the external address $(T_n)$ to $(T_{n+1})$ is a *subshift of finite type* on the set

$$\mathcal{A}^{\mathbb{N}_0} = (\mathcal{A}^\infty_0 \times \mathcal{A}^{\mathbb{N}}) \sqcup (\mathcal{A}^\infty_\infty \times \mathcal{A}^{\mathbb{N}}) \sqcup (\mathcal{A}^0_0 \times \mathcal{A}^{\mathbb{N}}) \sqcup (\mathcal{A}^0_\infty \times \mathcal{A}^{\mathbb{N}}),$$

where, if $e_0, e_1 \in \{0, \infty\}$, the set $\mathcal{A}^{e_1}_{e_0} \times \mathcal{A}^{\mathbb{N}}$ consists of the sequences in $\mathcal{A}^{\mathbb{N}_0}$ whose first symbol is in $\mathcal{A}^{e_1}_{e_0}$. Observe that the transition graph of $\sigma$ is

$$\begin{tikzcd}
\mathcal{A}^\infty_0 \times \mathcal{A}^{\mathbb{N}} \arrow[r] \arrow[dr, shift left] & \mathcal{A}^\infty_\infty \times \mathcal{A}^{\mathbb{N}} \arrow[loop right] \arrow[d] \\
\mathcal{A}^0_0 \times \mathcal{A}^{\mathbb{N}} \arrow[loop left] \arrow[u] & \mathcal{A}^0_\infty \times \mathcal{A}^{\mathbb{N}} \arrow[l] \arrow[ul, shift left]
\end{tikzcd}$$

and, in particular, not all sequences in $\mathcal{A}^{\mathbb{N}_0}$ are external addresses of points in $J(F)$.

We now introduce the notion of admissible external address. Only admissible external addresses can be the external address of a point in $J(F)$.

**Definition 5.7** (Admissible external address)**.** We say that an external address $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ is *admissible* if $\underline{s}$ belongs to the set

$$\mathbf{\Sigma}_e := \prod_{n \in \mathbb{N}} \mathcal{A}^{e_{n+1}}_{e_n} = \{(T_n) \ : \ T_n \in \mathcal{A}^{e_{n+1}}_{e_n} \text{ for all } n \in \mathbb{N}_0\},$$

for some $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$. In this case, we say that the external address $\underline{s}$ has *essential itinerary* $e$. We denote by $\mathbf{\Sigma}$ the set of *all* admissible external addresses.

Note that, if we define

$$\mathcal{A}_0 := \mathcal{A}^0_0 \sqcup \mathcal{A}^\infty_0 \quad \text{ and } \quad \mathcal{A}_\infty := \mathcal{A}^0_\infty \sqcup \mathcal{A}^\infty_\infty,$$

then an external address $\underline{s} = (T_n) \in \mathbf{\Sigma}$ has essential itinerary $e = (e_n)$ provided that $T_n \in \mathcal{A}_0$ if and only if $e_n = 0$. In terms of essential itineraries, the corresponding transition graph is the complete graph on two vertices,

$$\begin{tikzcd}
\mathcal{A}_0 \times \mathcal{T}^{\mathbb{N}} \arrow[loop left] \arrow[r, shift left] & \mathcal{A}_\infty \times \mathcal{T}^{\mathbb{N}} \arrow[loop right] \arrow[l, shift left]
\end{tikzcd}.$$

If $F \in \mathcal{B}^{*n}_{log}$, then $z \in I(F)$ has essential itinerary $e$ if and only if $\operatorname{addr}(z)$ has essential itinerary $e$. However, if $F$ is not normalised, these two sequences may be different for a certain number of iterates (see Lemma 7.6).

For every admissible external address, we introduce the set of points that have that external address. Note that sometimes we use the term external address to denote a general sequence in $\mathbf{\Sigma}$, without being necessarily the external address of any point $z \in J(F)$. Therefore, some of the following sets may be empty.

**Definition 5.8** (Subsets of $J(F)$)**.** Let $F$ be a function in the class $\mathcal{B}^*_{log}$. If $\underline{s} \in \mathbf{\Sigma}$ and $K > 0$, we define the sets

$$J_{\underline{s}}(F) := \{z \in J(F) \ : \ \mathrm{addr}_F(z) = \underline{s}\},$$

$J^K_{\underline{s}}(F) := J_{\underline{s}}(F) \cap J^K(F)$ and $I_{\underline{s}}(F) := J_{\underline{s}}(F) \cap I(F)$. For $e \in \{0, \infty\}^{\mathbb{N}_0}$ and $K > 0$, we define the sets

$$J_e(F) := \{z \in J(F) \ : \ \mathrm{addr}_F(z) \in \mathbf{\Sigma}_e\} = \bigcup_{\underline{s} \in \mathbf{\Sigma}_e} J_{\underline{s}}(F),$$

$J^K_e(F) := J_e(F) \cap J^K(F)$ and $I_e(F) := J_e(F) \cap I(F)$. If $F$ is normalised, then $I_e(F) = J(F) \cap \exp^{-1} I^{0,0}_e(f)$.

There is a natural way to order the tracts with respect to the vertical position that they are attached to infinity. Using this, we can endow the set of sequences $\mathbf{\Sigma}_e$ with the lexicographic order.

**Definition 5.9** (Lexicographic order)**.** Let $F : \mathcal{T} \to H$ be a function in the class $\mathcal{B}^*_{log}$. If $T, T'$ are components of $\mathcal{T}_\infty$, then we say that $T < T'$ if $T'$ is in the *upper* connected component of the intersection of a right half-plane and the complement of $T$. If $T, T'$ are components of $\mathcal{T}_0$, then we say that $T < T'$ if $T'$ is in the *lower* connected component of the intersection of a left half-plane and the complement of $T$. Finally, if $\underline{s}, \underline{s}' \in \mathbf{\Sigma}_e$ for some $e \in \{0, \infty\}^{\mathbb{N}_0}$, we say that $\underline{s} < \underline{s}'$ if there is $k \in \mathbb{N}_0$ such that $T_n = T'_n$ for all $n < k$ and $T_k < T'_k$.

The set $\mathbf{\Sigma}_e$ endowed with the lexicographic order is a totally ordered space. Note that, since the map $F$ preserves the orientation, if $T_1 < T_2$ in $\mathcal{T}_\infty$ and $T$ is a component of $\mathcal{T}_0$, then with the lexicographic ordering we have $F^{-1}_T(T_1) < F^{-1}_T(T_2)$.

Sometimes it will be useful to consider a partition of the tracts into fundamental domains (see Figure 5). The following terminology was introduced in [39].

**Definition 5.10** (Fundamental domain)**.** Let $f \in \mathcal{B}^*$ and let $F : \mathcal{T} \to H$ be a logarithmic transform of $f$ that is in the class $\mathcal{B}^*_{log}$. Let $\delta \subseteq \mathbb{C}^* \setminus \overline{\mathcal{V}}$ be the curve joining zero to infinity from Theorem 3.5.
(i) The preimages $\exp^{-1} \delta$ define infinitely many *fundamental strips* $S_n$, $n \in \mathbb{Z}$. Every tract of $F$ is contained in a fundamental strip.
(ii) For each tract $T_n$ of $F$, the restriction $F_{|T_n} : T_n \to H$ is a one-to-one covering of either $H_0$ or $H_\infty$. Hence, the set $F^{-1}_{|T_n}\left(H \setminus \exp^{-1} \delta\right)$ has infinitely many components $F_{n,i} \subseteq T_n$, $i \in \mathbb{Z}$, that we call *fundamental domains* of $F$.
(iii) Similarly, the preimages $f^{-1}(\delta)$ divide each tract $V_n$ of $f$ into infinitely many sets $D_{n,i} = \exp F_{m,i} \subseteq V_n$, $i \in \mathbb{Z}$, for some $m \in \mathbb{Z}$, that we call *fundamental domains* of $f$.

Note that sometimes we will refer to a sequence of fundamental domains using only one subindex when we do not need to specify whether two fundamental domains are a subset of the same tract or not.

Since the orbit of every point in $J(F)$ avoids $\exp^{-1}(\delta)$, we can define external addresses in terms of fundamental domains rather than tracts. This is the approach followed, for example, by Benini and Fagella [4]. However, since the image of each

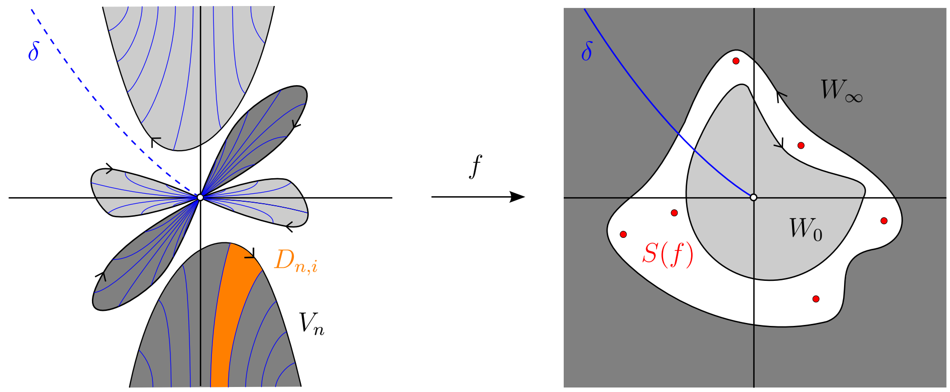


FIGURE 5. Fundamental domains of a function $f$ in the class $\mathcal{B}^*$.

fundamental domain is contained in a fundamental strip, the fundamental domain $F_n$ is determined by tract $T_n$ that contains $F_n$ and the fundamental strip containing the next tract $T_{n+1}$. Thus, considering external addresses of fundamental domains does not add more information to the symbolic dynamics of $F$.

We can also consider external addresses for functions $f \in \mathcal{B}^*$ rather than for their logarithmic transforms. In this case, specifying the sequence of tracts in $\mathcal{V}$ does not capture the whole combinatorics of $f$; we define the external addresses of $f$ in terms of fundamental domains. Let $\mathcal{A}_f$ denote the symbolic alphabet consisting of the fundamental domains of $f$.

**Definition 5.11** (External address of $f$)**.** Let $f \in \mathcal{B}^*$ and let $F \in \mathcal{B}^*_{log}$ be a periodic logarithmic transform of $f$. If $z = \exp w$, where $w \in J(F)$, we define the *external address* (under $f$) of $z$, $\operatorname{addr}_f(z)$, to be the symbol sequence $\underline{t} = (D_n) \in \mathcal{A}_f^{\mathbb{N}_0}$ such that $f^n(z) \in D_n$ for all $n \in \mathbb{N}_0$.

The next lemma describes the correspondence between external addresses of $f$ and external addresses of a logarithmic transform $F$ of $f$ (see [4, Lemma 2.9]).

**Lemma 5.12.** *Let $f \in \mathcal{B}^*$ and let $F \in \mathcal{B}^*_{log}$ be a logarithmic transform of $f$. If $z = \exp w$, then the external address $\operatorname{addr}_f(z) = (D_n)$ is uniquely determined by the external address $\operatorname{addr}_F(w) = (T_n)$. Conversely, if $\operatorname{addr}_f(z) = (D_n)$, then $\operatorname{addr}_F(w) = (T_n)$ is unique up to replacing $T_0$ by a $2k\pi i$-translate of $T_0$ for some $k \in \mathbb{Z}$ and changing the rest of $T_n$, $n > 0$, accordingly.*

*Proof.* Let $(T_n)$ be a sequence of tracts of $F$, then the sequence of fundamental domains $(D_n) \subseteq \mathcal{V}$ is given by $D_n = \exp F_n$ which, in turn, is determined by $T_n$ and $T_{n+1}$.

On the other hand, if $(D_n)$ is a sequence of fundamental domains of $f$, then the tract $T_0 \supseteq F_0$, where $\exp F_0 = D_0$, is given by the choice of the logarithmic transform $F$, which is unique up to addition of integer multiples of $2\pi i$, and the rest of tracts in the sequence $(T_n)$ satisfy that $T_n$ is the only tract in the fundamental strip $F(F_{n-1})$ containing a component of $\exp^{-1}(D_n)$. □

We say that a sequence of fundamental domains $(D_n)$ is *admissible* if it corresponds to an admissible external address $\underline{s} \in \mathbf{\Sigma}$. In this paper we use external addresses in terms of tracts mostly and restrict the use of fundamental domains to the moments when we need them, in order to keep the notation simpler.

6. **Unbounded continua in** $J_{\underline{s}}(F)$**.** A priori, the set $J_{\underline{s}}(F)$ may be empty for some external addresses in $\underline{s} \in \mathbf{\Sigma}$. Rippon and Stallard showed that, for a general transcendental entire function $f$, the components of the fast escaping set $A(f) \subseteq I(f)$, which was previously introduced by Bergweiler and Hinkkanen [8], are all unbounded [43]. Using similar ideas, Rempe showed that if $f \in \mathcal{B}$ (and the same argument works for class $\mathcal{B}_{log}$), then every tract $T$ contains an unbounded closed connected set $A$ consisting of points that escape within $T$ [39, Theorem 2.4]. Sometimes we refer to an unbounded closed connected set $X \subseteq \mathbb{C}$ as an *unbounded continuum*; note, however, that such set is not a continuum in $\mathbb{C}$ as it is not compact, but $X \cup \{\infty\}$ is a continuum in $\hat{\mathbb{C}}$ (see Lemma 6.2).

Although [39, Theorem 2.4] only concerns points that escape within a tract, if $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ is a periodic external address, then it follows that $J_{\underline{s}}(F)$ contains an unbounded continuum of escaping points. Indeed, if $\underline{s} = \overline{T_0 T_1 \dots T_{p-1}}$ has period $p \in \mathbb{N}$ and $T_k$, $0 \leqslant k < p$, are tracts of $F$, then there is a tract $T$ of $F^p$ contained in $T_0$ such that $F^k(T) \subseteq T_k$, $1 \leqslant k < p$, and the result follows from applying [39, Theorem 2.4] to $F^p$ in $T$.

It was remarked in [3, p. 2107] that if $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ contains only finitely many symbols, then [39, Theorem 2.4] can be adapted to show that $J_{\underline{s}}(F) \neq \emptyset$ and hence the set $J_{\underline{s}}(F)$ contains an unbounded continuum; see [4, Proposition 2.11] for a detailed proof of this result.

In [38], Rempe showed that the set $A$ can be chosen to be forward invariant. Later, Bergweiler, Rippon and Stallard [9, Theorem 1.1] generalised the result of Rempe for transcendental meromorphic functions in $\mathbb{C}$ with tracts (not necessarily in the class $\mathcal{B}$).

For transcendental self-maps of $\mathbb{C}^*$, we can define the fast escaping set $A(f)$ combining the iterates of the maximum and minimum modulus functions (see [30, Definition 1.2]), and then the components of $A(f)$ are unbounded in $\mathbb{C}^*$ [29, Theorem 1.5]. We recall that a set $X$ is *unbounded* in $\mathbb{C}^*$ if its closure $\widehat{X}$ in $\hat{\mathbb{C}}$ contains zero or infinity. The following lemma is a combination of [30, Theorem 1.1 and Theorem 1.5] and follows from the constructions in their proofs. Recall that $I_e^{0,0}(f) \subseteq I_e(f)$ is the set of escaping points whose essential itinerary is exactly $e$.

**Lemma 6.1.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$. For each $e = (e_n) \in \{0, \infty\}^{\mathbb{N}_0}$, there exists an unbounded closed connected set $A_e \subseteq I_e^{0,0}(f)$ which consists of fast escaping points and whose closure $\widehat{A}_e$ in $\hat{\mathbb{C}}$ contains the point zero or infinity corresponding to the value of $e_0$.*

Lemma 6.1 implies that the set $J_e(F)$ contains at least one unbounded component. The goal of this section is to show that, under certain hypotheses, the set $J_{\underline{s}}(F)$ contains an unbounded continuum. We begin by stating the boundary bumping theorem [34, Theorem 5.6] (see also [44, Theorem A.4]) which implies that if $X \subseteq \hat{\mathbb{C}}$ is a compact connected set containing zero or infinity and $E = X \cap \mathbb{C}^*$, then every component of $E$ is unbounded in $\mathbb{C}^*$.

**Lemma 6.2** (Boundary bumping theorem)**.** *Let $X$ be a non-empty compact connected metric space and let $E \subsetneq X$ be non-empty. If $C$ is a connected component of $E$, then $\partial C \cap \partial E \neq \emptyset$ (where boundaries are taken relative to $X$).*

First we show that if $J_{\underline{s}}^K(F) \neq \emptyset$ for sufficiently large $K > 0$, then the set $J_{\underline{s}}(F)$ contains an unbounded continuum. The following proposition is the analogue of [44, Lemma 3.3] for the class $\mathcal{B}_{log}^*$. We include the proof for completeness.

**Proposition 6.3.** *Let $F \in \mathcal{B}^*_{log}$. There exists $K_1(F) \geqslant 0$ such that if $K \geqslant K_1(F)$, for every $\underline{s} \in \mathbf{\Sigma}$, if $z_0 \in J^K_{\underline{s}}(F)$, then there exists an unbounded closed connected set $A \subseteq J_{\underline{s}}(F)$ with* $\operatorname{dist}(z_0, A) \leqslant 2\pi$.

*Proof.* We may assume without loss of generality that $F$ is normalised with $H = \mathbb{H}^{\pm}_R$ for some $R > 0$. Let $K_1(F) > 0$ be large enough that if $K \geqslant K_1(F)$, then all bounded components of $H \cap \overline{\mathcal{T}}$ are in the vertical band $V_K := \{z \in \mathbb{C} : |\operatorname{Re} z| < K\}$. Note that the set $V_K$ can only intersect a finite number of tracts in each fundamental strip.

Let $Y \subseteq \mathbb{C}$ be an unbounded continuum such that $Y \setminus D(F^k(z_0), 2\pi)$ has exactly one unbounded component. In that case we denote this component by $X_k(Y)$. Let $\underline{s} = (T_n) \in \mathbf{\Sigma}$. For all $k \geqslant 1$, we have that $\emptyset \neq X_k(\overline{T_k}) \subseteq H$ and hence $F^{-1}_{|T_{k-1}}$ maps $X_k(\overline{T_k})$ into $T_{k-1}$. By the expansivity property (5), since $\operatorname{dist}(F^k(z_0), X_k(T_k)) = 2\pi$, we have that $\operatorname{dist}(F^{k-1}(z_0), F^{-1}_{T_{k-1}}(X_k(T_k))) \leqslant \pi$ and $X_{k-1}(F^{-1}_{T_{k-1}}(X_k(T_k))) \neq \emptyset$. Thus, we can define the sets

$$A_k := X_0\big(F^{-1}_{T_0}\big(\cdots\big(X_{k-1}\big(F^{-1}_{T_{k-1}}\big(X_k(T_k)\big)\big)\big)\cdots\big)\big) \quad \text{ for } k \geqslant 1,$$

and we put $A_0 := X_0(\overline{T_0})$. Observe that here we are using the fact that $\underline{s} \in \mathbf{\Sigma}$ because $F^{-1}_{T_k}$ is only defined in one of the two components of $H$.

Let $\widehat{A}_k$ denote the closure of $A_k$ in $\hat{\mathbb{C}}$, which is a continuum. By construction, $\widehat{A}_{k+1} \subseteq \widehat{A}_k$ and $\operatorname{dist}(z_0, A_k) \leqslant \pi$, thus

$$A' := \bigcap_{k \geqslant 0} \widehat{A}_k$$

is a continuum in $\hat{\mathbb{C}}$ and $A' \setminus \{0, \infty\}$ has a component $A$ such that $\operatorname{dist}(z_0, A) \leqslant 2\pi$. Finally, by Lemma 6.2, the set $A$ is unbounded in $\mathbb{C}^*$ . $\square$

Next we show that, as in the entire case, if an external address $\underline{s} \in \mathbf{\Sigma}$ has only finitely many symbols, then the set $J_{\underline{s}}(F)$ contains an unbounded continuum. Note that in contrast to the previous proposition, now we need to show that $J_{\underline{s}}(F) \neq \emptyset$. We use the following lemma which is the analogue of [4, Proposition 2.6] for the class $\mathcal{B}^*$.

**Lemma 6.4.** *Let $F \in \mathcal{B}^*_{log}$ have good geometry and let $\mathcal{F}$ be a finite union of fundamental domains of $F$. Then, for any $K > 0$ sufficiently large,*

$$F^{-1}\big(\{z \in \mathbb{C} \ : \ |\operatorname{Re} z| = K\}\big) \cap \mathcal{F} \subseteq \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| < K\}.$$

In the following proposition we adapt the proof of [4, Proposition 2.11] to our setting. This is based on the ideas of [39, Theorem 2.4] and will be used later to prove Theorem 1.6.

**Proposition 6.5.** *Let $F \in \mathcal{B}^*_{log}$. There exists $K_2(f) > 0$ such that if $K \geqslant K_2(F)$ and $\underline{s} \in \mathbf{\Sigma}$ contains only finitely many different symbols, then $J^K_{\underline{s}}(F)$ contains a continuum whose points have unbounded real part.*

*Proof.* Suppose that $\underline{s} = (T_n)$ contains only $N \in \mathbb{N}$ different symbols corresponding to the tracts $T^s_1, \ldots, T^s_N$, and choose fundamental domains $F^s_{j,k} \subseteq T^s_j$, $1 \leqslant j \leqslant N$, so that $F(F^s_{j,k}) \supseteq T^s_k$. Let $\mathcal{F}$ denote the finite collection of fundamental domains $\{F^s_{j,k}\}_{1 \leqslant j \leqslant N}$ and assume $K_2 = K_2(F) > 0$ is sufficiently large that Lemma 6.4 holds for $\mathcal{F}$ and $K > K_2(F)$. Then define $(F_n)$ to be the sequence of fundamental domains from $\mathcal{F}$ satisfying that $F_n \subseteq T_n$ and $T_{n+1}$ lies in $F(F_n)$.

Let $X_0$ be the unbounded component of $F_0 \cap \mathbb{H}_K^{\pm}$ and, for each $n > 0$, let $X_n$ be the unique unbounded component of

$$F_{|F_0}^{-1}\big(\cdots\big(F_{|F_{n-2}}^{-1}\big(F_{|F_{n-1}}^{-1}(F_n) \cap \mathbb{H}_K^{\pm}\big) \cap \mathbb{H}_K^{\pm}\big)\cdots\big) \cap \mathbb{H}_K^{\pm},$$

where $F_{|F_n}^{-1}$ is the branch of $F^{-1}$ that maps the fundamental strip $F(F_n) \subseteq H$ in which $F_{n+1}$ lies to the fundamental domain $F_n \subseteq T_n$. Note that since $F$ is entire, $F_{|F_n}^{-1}$ maps unbounded sets to unbounded sets.

Lemma 6.4 tells us that $F^{-1}(\partial\mathbb{H}_K^{\pm}) \cap \mathcal{F} \subseteq \mathbb{C} \setminus \mathbb{H}_K^{\pm}$ and therefore for each $F_n \in \mathcal{F}$, necessarily $F_n \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$. Furthermore, if $Y$ is an unbounded continuum with $Y \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$, then, by Lemma 6.4, $F_{|F_n}^{-1}(Y) \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$. Thus, since $F_n \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$, we have that $X_n \cap \partial\mathbb{H}_K^{\pm} \neq \emptyset$ for all $n \in \mathbb{N}_0$.

As before, let $\widehat{X}_n$ be the closure of $X_n$ in $\hat{\mathbb{C}}$ and define

$$X' := \bigcap_{k \in \mathbb{N}_0} \widehat{X}_n,$$

which is an unbounded continuum. Since all the unbounded continua $\widehat{X}_n$ intersect the set $\partial\mathbb{H}_K^{\pm}$, $X' \setminus \{0, \infty\}$ has a component $X$ that intersects $\partial\mathbb{H}_K^{\pm}$ and is unbounded by Lemma 6.2. □

In particular, Proposition 6.5 covers all the periodic external addresses in $\mathbf{\Sigma}$. Observe that by considering external addresses that consist of fundamental domains instead of tracts we would obtain the result that for all such sequences containing only finitely many different fundamental domains of $f$ there is an unbounded continuum consisting of escaping points whose orbit lies in that sequence of fundamental domains.

7. **Dynamic rays.** In Theorem 1.1 we showed that bounded-type functions have no escaping Fatou components. Instead, escaping points often lie in curves tending to the essential singularities called *dynamic rays* or, sometimes, *hairs* such that in every unbounded proper subset of them, a *ray tail*, points escape uniformly.

**Definition 7.1** (Dynamic ray)**.** Let $f$ be a transcendental self-map of $\mathbb{C}^*$. A *ray tail* of $f$ is an injective curve

$$\gamma : [0, +\infty) \to I(f)$$

such that $f^n(\gamma(t)) \to \{0, \infty\}$ as $t \to +\infty$ for all $n \in \mathbb{N}_0$ and $f^n(\gamma(t)) \to \{0, \infty\}$ uniformly in $t$ as $n \to \infty$. A *dynamic ray* of $f$ is a maximal injective curve

$$\gamma : (0, +\infty) \to I(f)$$

such that $\gamma|_{[t,+\infty)}$ is a ray tail for every $t > 0$. Similarly, we can define ray tails for any logarithmic transform $F$ of $f$, which is only defined on the set $\mathcal{T}$, and dynamic rays for any lift $\tilde{f}$ of $f$. We shall abuse the notation and use $\gamma$ for both the ray as a set and its parametrization.

We say that a dynamic ray $\gamma$ is *broken* if $f^n(\gamma)$ contains a critical point for some $n \in \mathbb{N}_0$ (see [4, Definition 2.2]). A non-broken ray $\gamma$ is said to *land* if $\overline{\gamma} \setminus \gamma$ consists of a single point or, in other words, if $\gamma(t)$ has a limit as $t \to 0$.

**Example 7.2.** We give a couple of straightforward examples of dynamic rays in $\mathbb{C}^*$.

(i) The positive real line is a fixed dynamic ray for the map $f(z) = \exp(z + 1/z)$, and points there escape to infinity under iteration by $f$. This is an example of a broken ray because the function $f$ has a critical point at $z = 1$.

(ii) If we now consider the function $g(z) = \exp(-z + 1/z)$, the positive real line is again forward invariant but $z = 1$ is a repelling fixed point of $g$. In this case, the intervals $(0, 1)$ and $(1, +\infty)$ form a cycle of 2-periodic non-broken dynamic rays.

Observe that dynamic rays can land at an essential singularity and the limits of $\gamma(t)$ as $t \to 0$ and $t \to +\infty$ may coincide. The dynamic ray from the following example is non-broken and goes from zero to infinity.

**Example 7.3.** The positive real line is a fixed and non-broken dynamic ray for the function $f(z) = z \exp\bigl(z^2 + \exp(-1/z^2)\bigr)$ (see Figure 6).

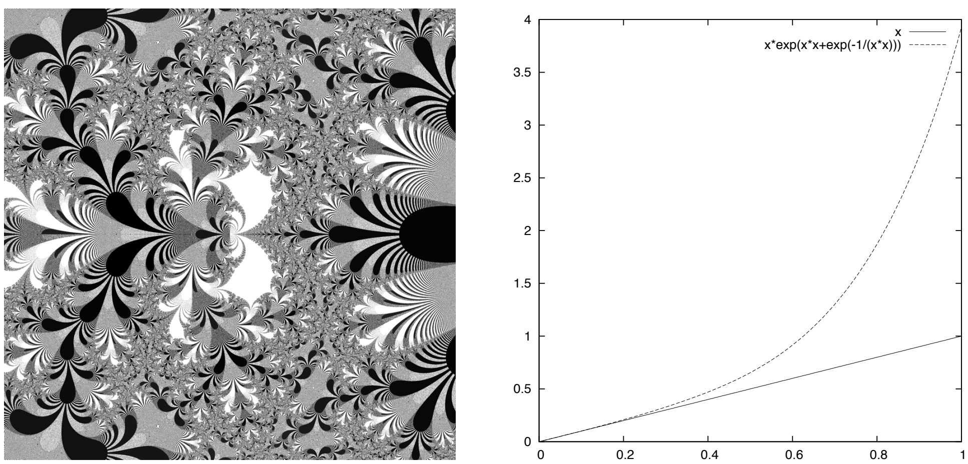


FIGURE 6. In the left, we have the phase space of the function $f(z) = z \exp(z^2 + \exp(-1/z^2))$ from Example 7.3. In the right, the graph of the restriction of this function to the positive real line.

Since the exponential function is a local homeomorphism, we have the following correspondence between dynamic rays of transcendental self-maps of $\mathbb{C}^*$ and those of their lifts.

**Lemma 7.4.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ and let $\tilde{f}$ be a lift of $f$. Then $\gamma$ is a dynamic ray of $f$ if and only if any connected component $\tilde{\gamma}$ of $\exp^{-1}\gamma$ is a dynamic ray of $\tilde{f}$. Furthermore, $\gamma$ lands or is broken if and only if $\tilde{\gamma}$ lands or is broken, respectively.*

It is a well-known result for transcendental entire functions that if the postsingular set is bounded, then all periodic dynamic rays land. This was first proved for the exponential family [46, 37]. Rempe proved a more general version of the result for Riemann surfaces that applies to maps in the classes $\mathcal{B}$ and $\mathcal{B}^*$ [39, Theorem B.1]; see also [12, Theorem 1.1] for an alternative proof of this result for the class $\mathcal{B}$. The same techniques imply the following result in our setting.

**Proposition 7.5.** *Let $f \in \mathcal{B}^*$ with postsingular set $P(f)$ bounded away from zero and infinity. Then all periodic dynamic rays of $f$ land, and the landing points are either repelling or parabolic periodic points of $f$.*

Next we show that, since points in ray tails escape uniformly, each dynamic ray is contained in a set $I_e(f)$ for some essential itinerary $e \in \{0, \infty\}^{\mathbb{N}_0}$.

**Lemma 7.6.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ and let $\gamma$ be a dynamic ray of $f$. Then, for every ray tail $\gamma' \subseteq \gamma$, there exists $\ell \in \mathbb{N}_0$ such that all the points in the curve $f^\ell(\gamma')$ have the same essential itinerary. Hence, $\gamma \subseteq I_e(f)$ for some sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$.*

*Proof.* By definition, ray tails escape uniformly and hence, if $\gamma'$ is a ray tail, there is $\ell \in \mathbb{N}_0$ such that $f^n(\gamma') \cap \mathbb{S}^1 = \emptyset$ for all $n \geqslant \ell$. Then, all points in $f^\ell(\gamma')$ have the same essential itinerary; that is, $\gamma' \subseteq I_e^{\ell,0}(f)$ for some $e \in \{0, \infty\}^{\mathbb{N}_0}$.

Now suppose that $\gamma$ is a dynamic ray with $z_1 \in \gamma \cap I_{e_1}(f)$ and $z_2 \in \gamma \cap I_{e_2}(f)$. Then there is a ray tail $\gamma' \supseteq \{z_1, z_2\}$ and $\ell \in \mathbb{N}_0$ such that all points in $f^\ell(\gamma')$ have the same essential itinerary. Thus, $e_1 \cong e_2$ and $\gamma \subseteq I_{e_1}(f) = I_{e_2}(f)$. □

Actually, since all the iterates of a dynamic ray are unbounded in $\mathbb{C}^*$, dynamic rays are asymptotically contained in tracts, which are preimages of the neighbourhood $W$ of the set $\{0, \infty\}$. Furthermore, each dynamic ray is asymptotically contained in exactly one of the fundamental domains of the function $F$.

In the following proposition we show that, in order to prove Theorem 1.4, we only require that every escaping point has an iterate that is on a ray tail (see [44, Proposition 2.3]).

**Proposition 7.7.** *Let $f$ be a transcendental self-map of $\mathbb{C}^*$ and let $z \in I(f)$. Suppose that some iterate $f^k(z)$, $k \in \mathbb{N}_0$, is on a ray tail $\gamma_k$ of $f$. Then either $z$ is on a ray tail, or there is some $n \leqslant k$ such that $f^n(z)$ belongs to a ray tail that contains an asymptotic value of $f$.*

*Proof.* Suppose that $\gamma_k : [0, \infty) \to \mathbb{C}^*$ is a parametrization of such a ray tail and $\gamma_k(0) = f^k(z)$. Let $\gamma_{k-1} : [0, T) \to \mathbb{C}^*$ be a maximal lift of $\gamma_k$ satisfying that $\gamma_{k-1}(0) = f^{k-1}(z)$ and $f(\gamma_{k-1}(t)) = \gamma_k(t)$ for $t \in [0, T)$. If $T = +\infty$, then $\gamma_{k-1}(t)$ must tend to zero or infinity as $t \to +\infty$, otherwise we would have $\gamma_{k-1}(t) \to a \in \mathbb{C}^*$ as $t \to +\infty$, so

$$f(a) = f\left(\lim_{t\to+\infty} \gamma_{k-1}(t)\right) = \lim_{t\to+\infty} f\left(\gamma_{k-1}(t)\right) = \lim_{t\to+\infty} \gamma_k(t) \in \{0, \infty\},$$

which is a contradiction. Thus, $f^{k-1}(z)$ is on a ray tail. Now consider the case that $T < +\infty$ and let

$$w := \lim_{t\to T} \gamma_{k-1}(t) \in \hat{\mathbb{C}}.$$

Again, it cannot happen that $f(w) \in \{0, \infty\}$ because $\gamma_k(T)$ would be an asymptotic value, so $f(w) = \gamma_k(t_0)$ for some $t_0 \in [0, +\infty)$. In this case, $\gamma_{k-1}$ could be extended, contradicting its maximality. Note that if $w$ was a critical point we would need to choose a branch of $f^{-1}$. Thus, $w \in \{0, \infty\}$ and $\gamma_k(T)$ is an asymptotic value of $f$ (possibly zero or infinity). Then either we have a ray tail $\gamma_{k-1} \subseteq f^{-1}(\gamma_k) \subseteq I(f)$ connecting $f^{k-1}(z)$ to one of the essential singularities or $\gamma_k$ contains an asymptotic value. The result follows by applying the above reasoning inductively. □

Note that Proposition 7.7 can also be proved by applying its version for entire functions to a lift $\tilde{f}$ of $f$ and then use the correspondence from Lemma 7.4.

We conclude this section by stating a result about escaping points that follows from the expansivity property (5) in Lemma 3.6 (see [44, Lemma 3.2] for the analogue result on entire functions).

**Lemma 7.8.** *Let $F:\mathcal{T}\to H$ be a function in the class $\mathcal{B}^{*n}_{log}$ with $H=\mathbb{H}^{\pm}_R$ for some $R>0$. If $z,w\in J_{\underline{s}}(F)$ for some external address $\underline{s}\in\mathbf{\Sigma}$ and $z\neq w$, then*

$$\lim_{k\to+\infty}\max\{|\operatorname{Re}F^k(z)|,\ |\operatorname{Re}F^k(w)|\}=+\infty. \tag{9}$$

Observe that (9) does not imply that neither the point $z$ nor $w$ escape because both points may have an unbounded orbit but with a subsequence where their iterates are bounded. In the next section, we will introduce a condition for the function $F$ (see Definition 8.1) which implies that, in the situation of Lemma 7.8, both points $z$ and $w$ escape, and hence all points in $J_{\underline{s}}(F)$ except possibly one must escape.

Lemma 3.9, Lemma 7.8 and Proposition 6.3 correspond respectively to Lemma 3.1, Lemma 3.2 and Theorem 3.3 in [44, Section 3] and constitute the main tools to prove Theorem 1.4 in the next section.

8. **Proof of Theorem 1.4.** In this section we adapt the results in [44, Sections 4 and 5] to our setting. Since the proof of Theorem 1.4 follows closely that of [44, Theorem 1.2], we only sketch it and emphasize the differences between them.

The head-start condition is designed so that every escaping point is mapped eventually to a ray tail and hence we are able to apply Proposition 7.7 and conclude that either the point itself is in a ray tail or some iterate of it is in a ray tail that contains a singular value.

**Definition 8.1** (Head-start condition)**.** Let $F:\mathcal{T}\to H$ be a function in the class $\mathcal{B}^*_{log}$. We first define the *head-start condition* for tracts, then for external addresses and finally for logarithmic transforms.

- Let $T,T'$ be two tracts in $\mathcal{T}$ and let $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a (not necessarily strictly) monotonically increasing continuous function with $\varphi(x)>x$ for all $x\in\mathbb{R}_+$. We say that the pair $(T,T')$ satisfies the *head-start condition* for $\varphi$ if, for all $z,w\in\overline{T}$ with $F(z),F(w)\in\overline{T'}$,
$$|\operatorname{Re}w|>\varphi(|\operatorname{Re}z|)\Rightarrow|\operatorname{Re}F(w)|>\varphi(|\operatorname{Re}F(z)|).$$
- We say that an external address $\underline{s}=(T_n)\in\mathbf{\Sigma}$ satisfies the *head-start condition* for $\varphi$ if all consecutive pairs of tracts $(T_k,T_{k+1})$ satisfy the head-start condition for $\varphi$, and if for all distinct $z,w\in J_{\underline{s}}(F)$, there is $M\in\mathbb{N}$ such that $|\operatorname{Re}F^M(z)|>\varphi(|\operatorname{Re}F^M(w)|)$ or $|\operatorname{Re}F^M(w)|>\varphi(|\operatorname{Re}F^M(z)|)$.
- We say that $F$ satisfies a *head-start condition* if every external address of $F$ satisfies the head-start condition for some $\varphi$. If the same function $\varphi$ can be chosen for all external addresses, we say that $F$ satisfies the uniform head-start condition for $\varphi$.

Notice that in the second part we require that the head-start condition cannot be a void condition for any itinerary. Furthermore, if $|\operatorname{Re}F^M(z)|>\varphi(|\operatorname{Re}F^M(w)|)$ and the head-start condition is satisfied for all consecutive pairs of tracts $(T_n,T_{n+1})$ for all $n\geqslant M$, then we have $|\operatorname{Re}F^n(z)|>\varphi(|\operatorname{Re}F^n(w)|)$ for all $n>M$.

The head-start condition allows us to order the points in $J_{\underline{s}}(F)$ by the growth of the absolute value of their real parts.

**Definition 8.2** (Speed ordering)**.** Let $F\in\mathcal{B}^*_{log}$ and let $\underline{s}\in\mathbf{\Sigma}$ be an external address satisfying the head-start condition for a function $\varphi$. For $z,w\in J_{\underline{s}}(F)$, we say that $z\succ w$ if there exists $K\in\mathbb{N}_0$ such that $|\operatorname{Re}F^K(z)|>\varphi(|\operatorname{Re}F^K(w)|)$. We extend this order to the closure $\widehat{J}_{\underline{s}}(F)$ in $\hat{\mathbb{C}}$ by the convention that $0, \infty \succ z$ for all points $z \in J_{\underline{s}}(F)$.

Note that although a dynamic ray may contain both zero and infinity in its closure in $\hat{\mathbb{C}}$, ray tails are a subset of $\mathcal{T}$ and hence their closure in $\hat{\mathbb{C}}$ contains either zero or infinity.

The head-start condition implies that the speed ordering is a total order on the set $\widehat{J}_{\underline{s}}(F)$: if there were $M_1, M_2 \in \mathbb{N}_0$ such that $|\operatorname{Re} F^{M_1}(z)| > \varphi(|\operatorname{Re} F^{M_1}(w)|)$ and $|\operatorname{Re} F^{M_2}(w)| > \varphi(|\operatorname{Re} F^{M_2}(z)|)$ then we would get a contradiction because once we are in one of these situations and the head-start condition is satisfied then it is preserved by iteration, that is, for example, if $|\operatorname{Re} F^{M_1}(z)| > \varphi(|\operatorname{Re} F^{M_1}(w)|)$, then $|\operatorname{Re} F^{n}(z)| > \varphi(|\operatorname{Re} F^{n}(w)|)$ for all $n > M_1$. Therefore $z \succ w$ if and only if there exists $n_0 \in \mathbb{N}_0$ such that $|\operatorname{Re} F^{n}(z)| > |\operatorname{Re} F^{n}(w)|$ for all $n > n_0$, and hence the speed ordering does not depend on the choice of the function $\varphi$.

**Lemma 8.3.** *Let $F \in \mathcal{B}^*_{log}$ and let $\underline{s} \in \mathbf{\Sigma}_e$, $e \in \{0, \infty\}^{\mathbb{N}_0}$, be an external address that satisfies the head-start condition for a function $\varphi$. Then the order topology induced by the speed ordering $\succ$ on $\widehat{J}_{\underline{s}}(F)$ coincides with the topology as a subset of $\hat{\mathbb{C}}$ and, in particular, every connected component of $\widehat{J}_{\underline{s}}(F)$ is an arc.*

*Moreover, there exists $K' > 0$ independent of $\underline{s}$ such that $J^{K'}_{\underline{s}}(F)$ is either empty or contained in the unique unbounded component of $J_{\underline{s}}(F)$, which is an arc to the essential singularity $e_0$ all of whose points escape except possibly its finite endpoint.*

*Proof.* The first part follows from the fact that the map $\mathrm{id} : \widehat{J}_{\underline{s}}(F) \to (\widehat{J}_{\underline{s}}(F), \prec)$ is an homeomorphism (see [44, Theorem 4.4]). Indeed, for all $a \in \widehat{J}_{\underline{s}}(F)$, the sets

$$(a, +\infty)_{\prec} := \{z \in \widehat{J}_{\underline{s}}(F) \ : \ a \prec z\}, \quad (-\infty, a)_{\prec} := \{z \in \widehat{J}_{\underline{s}}(F) \ : \ z \prec a\},$$

are open sets in $\widehat{J}_{\underline{s}}(F)$ with the subspace topology of $\hat{\mathbb{C}}$: let $k \in \mathbb{N}_0$ be minimal with the property that $|\operatorname{Re} F^{k}(a)| > \varphi(|\operatorname{Re} F^{k}(z)|)$ then, by continuity, this inequality holds in a neighbourhood of $z$. Since $\widehat{J}_{\underline{s}}(F)$ with the order topology is Hausdorff, the map $\mathrm{id}^{-1}$ is continuous as well. The theorem follows from the order characterisation of the arc (see [44, Theorem A5]).

For the second part, if $K \geqslant K_1(F)$, where $K_1(F) \geqslant 0$ is the constant from Proposition 6.3, and $J^{K}_{\underline{s}}(F) \neq \emptyset$, then $J^{K}_{\underline{s}}(F)$ has an unbounded component $A$ which is an arc to infinity. Since $e_0$ is the largest element of $\widehat{J}_{\underline{s}}(F)$ in the speed ordering, the set $\widehat{J}_{\underline{s}}(F)$ has only one unbounded component. Using the head-start condition, it can be shown that if $z, w \in J_{\underline{s}}(F)$ and $w \succ z$, then $w \in I_{\underline{s}}(F)$ (see [44, Corollary 4.5]). Finally, the fact that $J^{K'}_{\underline{s}}(F) \subseteq A$ for some $K' > K$ follows from the expansivity of $F$ (see [44, Proposition 4.6]). $\square$

As in the entire case, the following theorem can be deduced from Lemma 8.3 (see [44, Theorem 4.2]).

**Theorem 8.4.** *Let $F \in \mathcal{B}^*_{log}$ satisfy a head-start condition. Then, for every $z \in I(F)$, there exists $k \in \mathbb{N}_0$ such that $F^k(z)$ is on a ray tail $\gamma$. This ray tail is the unique arc in $J(F)$ connecting $F^k(z)$ to either zero or infinity (up to reparametrization).*

Observe that Theorem 8.4 together with Proposition 7.7 imply that if $f$ is a transcendental self-map of $\mathbb{C}^*$ and $z \in I(f)$, then either $z$ is on a ray tail, or there is some $n \leqslant k$ such that $f^n(z)$ belongs to a ray tail that contains an asymptotic value of $f$.

Previously we have seen that if $f$ has finite order, then any logarithmic transform $F$ of $f$ has good geometry in the sense of Definition 3.13. To complete the proof of Theorem 1.4, we show that functions of good geometry satisfy a head-start condition.

**Theorem 8.5.** *Let $F \in \mathcal{B}^{*n}_{log}$ be a function with good geometry. Then $F$ satisfies a linear head-start condition.*

*Proof.* Let $\underline{s} \in \mathbf{\Sigma}$ be an external address and suppose that $F$ has bounded slope with constants $(\alpha, \beta)$. Then the orbits of any two points $z, w \in J_{\underline{s}}(F)$ eventually separate far enough one from the other. More precisely, if $K \geqslant 1$, there exists a constant $\delta = \delta(\alpha, \beta, K) > 0$ such that if $|z-w| \geqslant \delta$, then either

$$|\operatorname{Re} F^n(z)| > K|\operatorname{Re} F^n(w)| + |z-w| \quad \text{ or } \quad |\operatorname{Re} F^n(w)| > K|\operatorname{Re} F^n(z)| + |z-w|,$$

holds for all $n \geqslant 1$ (see [44, Lemma 5.2]). Hence the external address $\underline{s}$ satisfies the second part of the head-start condition with the linear function $\varphi(x) = Kx + \delta$.

It remains to check that if $\underline{s} = (T_n)$, then for all $k \in \mathbb{N}_0$ and for all $z, w \in T_k$ such that $F(z), F(w) \in T_{k+1}$, we have

$$|\operatorname{Re} w| > K|\operatorname{Re} z| + \delta \quad \Rightarrow \quad |\operatorname{Re} F(w)| > K|\operatorname{Re} F(z)| + \delta.$$

We omit the technical computations from this proof, which are identical to the ones for the entire case, and just observe that this follows from the fact that the tracts of $F$ have uniformly bounded wiggling with constants $K$ and $\mu$ for some $\mu > 0$ if and only if the conditions

$$|\operatorname{Re} w| > K|\operatorname{Re} z| + M'$$
$$|\operatorname{Im} F(z) - \operatorname{Im} F(w)| \leqslant \alpha \max\{|\operatorname{Re} F(z)|, |\operatorname{Re} F(w)|\} + \beta$$

imply that $|\operatorname{Re} F(w)| > K|\operatorname{Re} F(z)| + M'$ whenever $z, w \in T$, for some $M' > 0$. Hence $F$ satisfies the uniform linear head-start condition with constants $K$ and $M$ for some $M > 0$ (see [44, Proposition 5.4]). $\square$

Finally we prove Theorem 1.4 concerning the existence of dynamic rays for compositions of finite order transcendental self-maps of $\mathbb{C}^*$.

*Proof of Theorem 1.4.* Let $f_1, \dots, f_n$ be finite order transcendental self-maps of $\mathbb{C}^*$ for some $n \geqslant 1$. By Theorem 1.3, the functions $f_i$ are in class $\mathcal{B}^*$. Composing the functions $f_i$ with affine changes of variable, we can assume that each $f_i$ has a normalised logarithmic transform $F_i : \mathcal{T}_i \to \mathbb{H}^{\pm}_{R_i} \in \mathcal{B}^{*n}_{log}$ for some $R_i > 0$.

By Proposition 4.7, each $F_i$ has good geometry and hence, by Theorem 8.5, they satisfy linear head-start conditions. Just as for functions in $\mathcal{B}_{log}$, linear head-start conditions are preserved by composition in $\mathcal{B}^*_{log}$ (see [44, Lemma 5.7]). If the function $F_1$ has bounded slope and all $F_i$ satisfy uniform linear head-start conditions, then the function $F := F_n \circ \cdots \circ F_1 \in \mathcal{B}^*_{log}$, which is a logarithmic transform of $f = f_n \circ \cdots \circ f_1 \in \mathcal{B}^*$, has bounded slope and satisfies a uniform linear head-start condition when restricted to a suitable set of tracts.

Finally, we can apply Theorem 8.4 and Proposition 7.7 to conclude that every point $z \in I(f)$ is on a ray tail that joins $z$ to either zero or infinity. $\square$

**Remark 8.6.** The proof of Theorem 1.4 relies on normalised logarithmic transforms. However, it is possible to carry on the same ideas using only disjoint-type functions, so that the resulting function $F$ is also of disjoint type (see [44, Theorem 5.10] and [2, Theorem C]).

We conclude this section with the proof of Corollary 1.5, which follows easily from Theorem 1.4.

*Proof of Corollary 1.5.* Every function of the form $f(z) = nz + P(e^z) + Q(e^{-z})$ with $n \in \mathbb{Z}$ and $P, Q$ polynomials is a lift of a transcendental self-map of $\mathbb{C}^*$ of finite order. If $|\operatorname{Re} f^n(z)| \to +\infty$ as $n \to \infty$, then the point $z' = e^z$ lies in the escaping set of such self-map of $\mathbb{C}^*$. Thus, by Theorem 1.4, we can join $z'$ to either zero or infinity by a ray tail $\gamma$ and hence $z$ can be joined to infinity by curve in $\exp^{-1}\gamma$. □

**Remark 8.7.** The fact that a lift $\tilde{f}$ of a transcendental self-map $f$ of $\mathbb{C}^*$ has finite order does not imply that $f$ has finite order in $\mathbb{C}^*$. For instance, we may use a result by Clunie and Kövari [11] to construct a transcendental entire function $g$ of order zero such that $g \circ \exp$ has finite order and $\exp \circ g$ has infinite order.

9. **Periodic rays and Cantor bouquets.** In Section 6 we observed that the set $J_{\underline{s}}(F)$ may be empty for some $\underline{s} \in \mathbf{\Sigma}$. For transcendental entire functions in the exponential family, $f_\lambda(z) = \lambda e^z$, $\lambda \neq 0$, there is a characterization of which external addresses give rise to hairs, and this led to the notion of *exponentially bounded* (or *admissible*) external addresses in that context (see [45]). In particular, every periodic external address is exponentially bounded. Observe that the term admissible has a different meaning in this paper.

Barański, Jarque and Rempe [3] studied the set of dynamic rays for the functions considered in [44] and [2] and showed that they have uncountably many rays organised in a Cantor bouquet (see Definition 9.1). In this section we adapt their techniques to study the set of dynamic rays constructed in Section 8.

We begin by proving Theorem 1.6, which states that if $f \in \mathcal{B}^*$ satisfies the hypothesis of Theorem 1.4 and $\underline{t} = (D_n)$ is an admissible external address of $f$ which contains only finitely many symbols, then $f$ has a unique (non-empty) dynamic ray with external address $\underline{t}$. Furthermore, if $\underline{t}$ is periodic and the postsingular set $P(f)$ is bounded, then the dynamic ray lands.

*Proof of Theorem 1.6.* By Proposition 6.5, there exists an unbounded continuum $A \subseteq \mathcal{V}$ of escaping points with external address $\underline{t} = (D_n)$. Let $F$ be a periodic logarithmic transform of $f$, and let $\underline{s} = (T_n)$ be the external address that corresponds to the sequence of fundamental domains $(D_n)$ of $f$ by Lemma 5.12. By Theorem 1.4, the set $J_{\underline{s}}(F)$ is a dynamic ray $\tilde{\gamma}$, and the projection $\gamma = \exp \tilde{\gamma}$ is a dynamic ray of $f$ with external address $\underline{t}$. Finally, by Lemma 7.5, since $P(f)$ is bounded, all periodic rays land. □

Theorem 1.6 implies, for example, that each fundamental domain $D$ of $f$ contains exactly one fixed ray because the constant external address $(D_n)$ with $D_n = D$ for all $n \in \mathbb{N}_0$ is unique.

In Lemma 6.1, which summarized some results from [30], we saw that if $f$ is any transcendental self-map of $\mathbb{C}^*$ and $e \in \{0, \infty\}^{\mathbb{N}_0}$, then the set $I_e^{0,0}(f)$ contains an unbounded closed connected subset $A_e$. Furthermore, if $f \in \mathcal{B}^*$ and satisfies the hypotheses of Theorem 1.4, then Theorem 1.6 implies that the set $I_e^{0,0}(f)$ contains a ray tail; note that a dynamic ray may intersect the unit circle and hence contain points that are not in $I_e^{0,0}(f)$. Therefore, in this case, since the set $\{0, \infty\}^{\mathbb{N}_0}$ has uncountably many non-equivalent sequences $e$ and two such sequences give disjoint sets $I_e(f)$, the escaping set $I(f)$ contains uncountably many dynamic rays.

As explained in the introduction, a stronger result is true, namely Theorem 1.7, which states that for every essential itinerary $e \in \{0,\infty\}^{\mathbb{N}_0}$, the set $I_e^{0,0}(f)$ contains a *Cantor bouquet* and, in particular, uncountably many hairs. With the goal in mind of proving this theorem, we start by giving a precise definition of a Cantor bouquet (see [1, Definition 1.2]).

**Definition 9.1** (Cantor bouquet)**.** A set $B \subseteq [0,+\infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is called a *straight brush* if the following properties are satisfied:

(a) The set $B$ is a closed subset of $\mathbb{R}^2$.
(b) For every $(x,y) \in B$, there exists $t_y \geqslant 0$ such that $\{x : (x,y) \in B\} = [t_y,+\infty)$.
(c) The set $\{y : (x,y) \in B \text{ for some } x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every point $(x,y) \in B$, there exist two sequences of hairs attached, respectively, at $\beta_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \gamma_n$ for all $n \in \mathbb{N}$, and $\beta_n, \gamma_n \to y$ and $t_{\beta_n}, t_{\gamma_n} \to t_y$ as $n \to \infty$.

The set $[t_y,+\infty) \times \{y\}$ is called the *hair attached at* $y$ and the point $(t_y, y)$ is called its *endpoint*. A *Cantor bouquet* is a set $X \subseteq \mathbb{C}$ that is ambiently homeomorphic to a straight brush.

First we are going to show that, for each sequence $e \in \{0,\infty\}^{\mathbb{N}_0}$, the set $J(F)$ contains an *absorbing set* $X_e$ consisting of hairs such that every point in the set $I_e(F)$ enters $X_e$ after finitely many iterations (see [44, Theorem 4.7]). Recall that, for $e \in \{0,\infty\}^{\mathbb{N}_0}$, we defined the set

$$J_e(F) := \{z \in J(F) \ : \ \operatorname{addr}_F(z) \in \boldsymbol{\Sigma}_e\} = \bigcup_{\underline{s} \in \boldsymbol{\Sigma}_e} J_{\underline{s}}(F).$$

It will be helpful to use the following notation: for each $e \in \{0,\infty\}^{\mathbb{N}_0}$, we define the set of sequences

$$\boldsymbol{\Sigma}_e^+ := \bigcup_{n \in \mathbb{N}} \sigma^n(\boldsymbol{\Sigma}_e)$$

and the set

$$J_e^+(F) := \{z \in J(F) \ : \ \operatorname{addr}_F(z) \in \boldsymbol{\Sigma}_e^+\} = \bigcup_{n \in \mathbb{N}} J_{\sigma^n(e)}(F)$$

which is forward invariant.

**Proposition 9.2.** *Suppose that $F \in \mathcal{B}_{log}^*$ is a function that satisfies a head-start condition. Then, for every $e \in \{0,\infty\}^{\mathbb{N}_0}$, there exists a closed subset $X_e \subseteq J_e^+(F)$ with the following properties:*

*(a) $F(X_e) \subseteq X_e$.*
*(b) The connected components of $X_e$ are closed arcs to infinity all of whose points except possibly of its endpoint escape.*
*(c) Every point in $I_e(F)$ enters the set $X_e$ after finitely many iterations.*

*If $F$ is of disjoint type, then we may choose $X_e = J_e^+(F)$ and if $F$ is $2\pi i$-periodic, then $X_e'$ can also be chosen to be $2\pi i$-periodic.*

*Proof.* Let $X_e'$ be the union of all unbounded components of the set $J_e(F)$, and define the set

$$X_e := \bigcup_{n \in \mathbb{N}} X'_{\sigma^n(e)}.$$

Since unbounded components of $J(F)$ map to unbounded components of $J(F)$ by $F$, we have $F(X_e') \subseteq X'_{\sigma(e)}$ and hence $X_e$ is forward invariant.

By Lemma 6.2, the closure $\widehat{X}_e$ in $\hat{\mathbb{C}}$ is the connected component of $J_e^+(F) \cup \{\infty\}$ that contains infinity and hence the set $X_e$ is closed. By Lemma 8.3, the set $X_e$ consists of arcs to infinity all of whose points except possibly of its endpoint escape.

Let $K' \geqslant 0$ be the constant from Lemma 8.3, independent of $\underline{s} \in \mathbf{\Sigma}$, so that the set $J_{\underline{s}}^{K'}(F)$ is either empty or contained in the unbounded component of $J_{\underline{s}}(F)$ which is contained in $X_e$ if $\underline{s} \in \mathbf{\Sigma}_e^+$. Then (c) follows from the fact that points in the set $I_e(F)$ enter $J_{\sigma^n(e)}^{K'}(F) \subseteq X_e$ for some $n \in \mathbb{N}$ after finitely many iterations.

Finally, recall from Definition 3.3 that functions in the class $\mathcal{B}_{log}^*$ are of the form $F : \mathcal{T} \to H_0 \sqcup H_\infty$, where the sets $H_0$ and $H_\infty$ contain, respectively, a left and a right half-plane. If $F$ is of disjoint type, then

$$J_e(F) \cup \{\infty\} = \bigcup_{\underline{s} \in \mathbf{\Sigma}_e} \bigcap_{n \in \mathbb{N}} \Big( F_{|T_0}^{-1}( \cdots F_{|T_{n-2}}^{-1}(F_{|T_{n-1}}^{-1}(\overline{H}_{e_n})) \cdots ) \cup \{\infty\} \Big)$$

which is a union of nested intersections of unbounded continua, hence every component of $J_e(F)$ is an unbounded continuum and we can choose $X_e = J_e(F)$. If $F$ is a $2\pi i$-periodic function, then the set $X'_e$ is also $2\pi i$-periodic. $\square$

Following [3], the strategy to prove Theorem 1.7 will be, for each sequence $e \in \{0, \infty\}^{\mathbb{N}_0}$, to compactify the space of admissible external addresses $\mathbf{\Sigma}_e$ by adding a *circle of addresses at infinity* to show that the set $X'_e$ (and hence $X_e$) contains a Cantor bouquet.

**Lemma 9.3.** *For every $e \in \{0, \infty\}^{\mathbb{N}_0}$, there exists a totally ordered set $\tilde{\mathcal{S}}_e \supseteq \mathbf{\Sigma}_e$, where the order on $\tilde{\mathcal{S}}_e$ agrees with the lexicographic order on $\mathbf{\Sigma}_e$, and such that*

*(a) with the order topology, the set $\tilde{\mathcal{S}}_e$ is homeomorphic to $\mathbb{R} \cup \{-\infty, +\infty\}$;*
*(b) the set $\mathbf{\Sigma}_e$ is dense in $\tilde{\mathcal{S}}_e$.*

The construction of the set $\tilde{\mathcal{S}}_e$ is achieved by defining *intermediate entries* of each set $\mathcal{T}_{e_0}^{e_1}$ with $e_0, e_1 \in \{0, \infty\}$, that is, symbols which correspond to entries in between pairs of adjacent tracts as well as to limits of sequences of tracts. We then add *intermediate external addresses* to the set $\mathbf{\Sigma}_e$, that is, finite sequences of the form $\underline{s} = T_0 T_1 \ldots T_{n-1} S_n$, where $T_j \in \mathcal{T}_{e_j}^{e_{j+1}}$, $0 \leqslant j < n$, and $S_n$ is an intermediate entry of the set $\mathcal{T}_{e_n}^{e_{n+1}}$. We refer to [3, Section 5] for the details.

We can define a topology on the set $\tilde{H}_e := \overline{H}_{e_0} \cup \tilde{\mathcal{S}}_e$ that agrees with the induced topology on $H$ and such that $\tilde{H}_e$ is homeomorphic to the closed unit disc. Then, in this topology, the closure $\tilde{X}_e$ of the set $X_e$ from Proposition 9.2 is a *comb*, a compactification of a straight brush, with the arc $\tilde{\mathcal{S}}_e$ as base.

**Definition 9.4** (Comb)**.** A *comb* is a continuum $X$ containing an arc $B$, called the *base* of the comb, such that

(a) the closure of every component of $X \setminus B$ is an arc with exactly one endpoint in the base $B$;
(b) the intersection of the closures of any two hairs is empty;
(c) the set $X \setminus B$ is dense in $X$.

The fact that a Cantor bouquet consists of uncountably many hairs comes from the fact that a perfect set is uncountable. We introduce now the concept of (one-sided) hairy arc, a comb where every hair is accumulated by other hairs.

**Definition 9.5** (Hairy arc)**.** A *hairy arc* is a comb with base $B$ and an order $\prec$ on $B$ such that if $b \in B$ and $x$ belongs to the hair attached at $b$, then there exist sequences $(x_n^+)$ and $(x_n^-)$, attached respectively at points $b_n^+, b_n^- \in B$, such that

$b_n^- \prec b \prec b_n^+$ and $x_n^-, x_n^+ \to x$ as $n \to \infty$. A *one-sided hairy arc* is a hairy arc with all its hairs attached to the same side of the base.

Given a straight brush, it is easy to see that we can add a base to it in order to obtain a hairy arc. Aarts and Oversteegen showed that one-sided hairy arcs (and, in particular, straight brushes) are ambiently homeomorphic, and hence the converse of the previous statement is also true [1, Theorem 4.1].

**Lemma 9.6.** *Let $X$ be a one-sided hairy arc with base $B$. Then $X \setminus B$ is ambiently homeomorphic to a straight brush.*

In order to show that $X_e$ contains a Cantor bouquet, we prove that every hair in $X_e'$ is accumulated by hairs of the same set from both sides. To do so, we adapt the proof of [3, Proposition 7.3].

**Proposition 9.7.** *Let $F : \mathcal{T} \to H$ be a $2\pi i$-periodic function in the class $\mathcal{B}_{log}^*$, and let $e \in \{0, \infty\}^{\mathbb{N}_0}$ and $\tau > 0$. Then there is $\tau' \geqslant \tau$ such that for every $z_0 \in J_e^{\tau'}(F)$, there exist sequences $z_n^-, z_n^+ \in J_e^{\tau}(F)$ such that $addr(z_n^-) < addr(z_0) < addr(z_n^+)$ for all $n \in \mathbb{N}$ and $z_n^-, z_n^+ \to z_0$ as $n \to \infty$.*

*Proof.* Let $R_0$ be the constant from Lemma 3.6 so that $\mathbb{H}_{R_0}^{\pm} \subseteq H$ and $|F'(z)| \geqslant 2$ for $|\operatorname{Re} z| \geqslant R_0$. Let $n \in \mathbb{N}$, and let $\varphi_n : H_{e_n} \to H_{e_0}$ be the branch of $F^{-n}$ that maps $F^n(z_0)$ to $z_0$. Set $\tau' := \max\{R_0, \tau\} + \pi$ and define

$$z_n^{\pm} := \varphi_n\big(F^n(z_0) \pm 2\pi i\big) \in J_e^{\tau}(F).$$

Then $\operatorname{addr}(z_n^-) < \operatorname{addr}(z_0) < \operatorname{addr}(z_n^+)$ for all $n \in \mathbb{N}$. Finally, since $F$ is expanding with respect to the Euclidean metric on $\mathbb{H}_{R_0}^{\pm}$, the maps $\varphi_n$ are contractions and $z_n^{\pm} \to z_0$ as $n \to \infty$. $\square$

Note that given any logarithmic transform $F$ of a function $f \in \mathcal{B}^*$ we can modify it to obtain a periodic logarithmic transform of $f$ by adding a suitable multiple of $2\pi i$ to $F$ on each of its tracts.

Finally we sketch the proof of Theorem 1.7. The main idea is to use the existence of a potential function $\rho$ that 'straightens' the brush $X_e'$ (see [3, Proposition 7.1]).

*Proof of Theorem 1.7.* Let $F \in \mathcal{B}_{log}^*$ be $2\pi i$-periodic and satisfy a uniform head-start condition and let $X_e'$ denote the union of the unbounded components of $J_e(F)$ as in Proposition 9.2. For each $e \in \{0, \infty\}^{\mathbb{N}_0}$, consider the set

$$Z_e := \{z \in X_e' \ : \ \rho\big(F^j(z)\big) \geqslant K \text{ for all } j \in \mathbb{N}_0\} \cup \tilde{\mathcal{S}}_e,$$

where $\rho$ is a $2\pi i$-periodic continuous function that is strictly increasing on the hairs and such that $\rho(z_n) \to +\infty$ if and only if $|\operatorname{Re} z_n| \to +\infty$. Then, there exists $R > 0$ sufficiently large so that $J_e^R(F) \subseteq Z_e \subseteq \tilde{X}_e$ and hence $Z_e$ is a comb. Then Proposition 9.7 together with the fact that $F$ satisfies a uniform head-start condition imply that $Z_e$ is a hairy arc and, by Lemma 9.6, the set $Z_e \setminus \tilde{\mathcal{S}}_e$ is ambiently homeomorphic to a straight brush. We can choose the set $X_e$ from Proposition 9.2 to be $2\pi i$-periodic and so both $J_e(F)$ and $\exp(J_e(F))$ contain an absorbing Cantor bouquet. Note that all the points in $\exp(J_e(F))$ belong to $I_e^{0,0}(f)$ except, possibly, the finite endpoints of the hairs.

Finally, if $F$ is of disjoint type, then the closure of $J_e(F)$ in $\tilde{H}_e$ is a one-sided hairy arc, and hence both $J_e(F)$ and $\exp(J_e(F))$ are Cantor bouquets. $\square$

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## REFERENCES

[1] J. M. Aarts and L. G. Oversteegen, The geometry of Julia sets, *Trans. Amer. Math. Soc.*, **338** (1993), 897–918.

[2] K. Barański, Trees and hairs for some hyperbolic entire maps of finite order, *Math. Z.,* **257** (2007), 33–59.

[3] K. Barański, X. Jarque and L. Rempe, Brushing the hairs of transcendental entire functions, *Topology Appl.*, **159** (2012), 2102–2114.

[4] A. M. Benini and N. Fagella, A separation theorem for entire transcendental maps, *Proc. Lond. Math. Soc. (3),* **110** (2015), 291–324.

[5] W. Bergweiler, Iteration of meromorphic functions, *Bull. Amer. Math. Soc. (N.S.)*, **29** (1993), 151–188.

[6] W. Bergweiler, On the Julia set of analytic self-maps of the punctured plane, *Analysis*, **15** (1995), 251–256.

[7] W. Bergweiler and A. E. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Rev. Mat. Iberoamericana*, **11** (1995), 355–373.

[8] W. Bergweiler and A. Hinkkanen, On semiconjugation of entire functions, *Math. Proc. Cambridge Philos. Soc.,* **126** (1999), 565–574.

[9] W. Bergweiler, P. J. Rippon and G. M. Stallard, Dynamics of meromorphic functions with direct or logarithmic singularities, *Proc. Lond. Math. Soc. (3),* **97** (2008), 368–400.

[10] C. J. Bishop, Constructing entire functions by quasiconformal folding, *Acta Math.,* **214** (2015), 1–60.

[11] J. Clunie and T. Kövari, On integral functions having prescribed asymptotic growth. II, *Canad. J. Math.,* **20** (1968), 7–20.

[12] A. Deniz, A landing theorem for periodic dynamic rays for transcendental entire maps with bounded post-singular set, *J. Difference Equ. Appl.*, **20** (2014), 1627–1640.

[13] R. L. Devaney and M. Krych, Dynamics of $\exp(z)$, *Ergodic Theory Dynam. Systems*, **4** (1984), 35–52.

[14] R. L. Devaney and F. Tangerman, Dynamics of entire functions near the essential singularity, *Ergodic Theory Dynam. Systems*, **6** (1986), 489–503.

[15] A. Douady and J. H. Hubbard, *Étude Dynamique Des Polynômes Complexes. Partie I/II*, Publications Mathématiques d'Orsay, **84/85**, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984/1985.

[16] A. E. Eremenko, On the iteration of entire functions, *Dynamical systems and ergodic theory (Warsaw, 1986)*, Banach Center Publ., PWN, Warsaw, **23** (1989), 339–345.

[17] A. E. Eremenko and M. Yu. Lyubich, Dynamical properties of some classes of entire functions, *Ann. Inst. Fourier (Grenoble)*, **42** (1992), 989–1020.

[18] N. Fagella, Dynamics of the complex standard family, *J. Math. Anal. Appl.*, **229** (1999), 1–31.

[19] P. Fatou, Sur l'itération des fonctions transcendantes entières, *Acta Math.*, **47** (1926), 337–370.

[20] O. Forster, *Lectures on Riemann Surfaces*, Translated from the German by Bruce Gilligan. Graduate Texts in Mathematics, **81**, Springer-Verlag, New York-Berlin, 1981.

[21] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[22] M. Heins, Entire functions with bounded minimum modulus; subharmonic function analogues, *Ann. of Math. (2)*, **49** (1948), 200–213.

[23] F. Iversen, *Recherches sur les fonctions inverses des fonctions méromorphes*, Ph.D. thesis, Helsingin Yliopisto, 1914.

[24] L. Keen, Dynamics of holomorphic self-maps of $\mathbf{C}^*$, *Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986)*, 9–30, Math. Sci. Res. Inst. Publ. **10**, Springer, New York, 1988.

[25] L. Keen, Topology and growth of a special class of holomorphic self-maps of $\mathbf{C}^*$, *Ergodic Theory Dynam. Systems*, **9** (1989), 321–328.

[26] J. Kotus, Iterated holomorphic maps on the punctured plane, *Dynamical systems (Sopron, 1985)*, 10–28, Lecture Notes in Econom. and Math. Systems **287**, Springer, Berlin, 1987.

[27] J. K. Langley, On the multiple points of certain meromorphic functions, *Proc. Amer. Math. Soc.*, **123** (1995), 1787–1795.

[28] P. M. Makienko, Iterations of analytic functions in $\mathbf{C}^*$, *Dokl. Akad. Nauk SSSR*, **297** (1987), 35–37; translation in *Soviet Math. Dokl.,* **36** (1988), 418–420.

[29] D. Martí-Pete, *Structural Theorems for Holomorphic Self-maps of the Punctured Plane*, Ph.D. thesis, The Open University, 2016.

[30] D. Martí-Pete, The escaping set of transcendental self-maps of the punctured plane, to appear in *Ergodic Theory Dynam. Systems*, `arXiv:1412.1032`.

[31] D. Martí-Pete, Escaping Fatou components of transcendental self-maps of the punctured plane, in preparation.

[32] H. Mihaljević-Brandt and L. Rempe-Gillen, Absence of wandering domains for some real entire functions with bounded singular sets, *Math. Ann.*, **357** (2013), 1577–1604.

[33] J. W. Milnor, *Dynamics in One Complex Variable*, $3^{rd}$ edition, Annals of Mathematics Studies, **160**, Princeton University Press, Princeton, NJ, 2006.

[34] S. B. Nadler, Jr., *Continuum Theory. An introduction*, Monographs and Textbooks in Pure and Applied Mathematics, **158**, Marcel Dekker Inc., New York, 1992.

[35] G. Pólya, On an integral function of an integral function, *J. London Math. Soc.*, **S1-1** (1925), 12–15.

[36] H. Rådström, On the iteration of analytic functions, *Math. Scand.*, **1** (1953), 85–92.

[37] L. Rempe, A landing theorem for periodic rays of exponential maps, *Proc. Amer. Math. Soc.*, **134** (2006), 2639–2648 (electronic).

[38] L. Rempe, On a question of Eremenko concerning escaping components of entire functions, *Bull. Lond. Math. Soc.*, **39** (2007), 661–666.

[39] L. Rempe, Siegel disks and periodic rays of entire functions, *J. Reine Angew. Math.,* **624** (2008), 81–102.

[40] L. Rempe, P. J. Rippon and G. M. Stallard, Are Devaney hairs fast escaping?, *J. Difference Equ. Appl.*, **16** (2010), 739–762.

[41] L. Rempe-Gillen and D. J. Sixsmith, Hyperbolic entire functions and the Eremenko-Lyubich class: Class $\mathcal{B}$ or not class $\mathcal{B}$?, to appear in *Math. Z.*, (2016), 1–18, `arXiv:1502.00492`.

[42] P. J. Rippon and G. M. Stallard, Dimensions of Julia sets of meromorphic functions, *J. London Math. Soc. (2)*, **71** (2005), 669–683.

[43] P. J. Rippon and G. M. Stallard, On questions of Fatou and Eremenko, *Proc. Amer. Math. Soc.*, **133** (2005), 1119–1126 (electronic).

[44] G. Rottenfusser, J. Rückert, L. Rempe and D. Schleicher, Dynamic rays of bounded-type entire functions, *Ann. of Math. (2)*, **173** (2011), 77–125.

[45] D. Schleicher and J. Zimmer, Escaping points of exponential maps, *J. London Math. Soc. (2)*, **67** (2003), 380–400.

[46] D. Schleicher and J. Zimmer, Periodic points and dynamic rays of exponential maps, *Ann. Acad. Sci. Fenn. Math.* ,**28** (2003), 327–354.

[47] D. J. Sixsmith, A new characterisation of the Eremenko-Lyubich class, *J. Anal. Math.*, **123** (2014), 95–105.

[48] D. Sullivan, Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains, *Ann. of Math. (2)*, **122** (1985), 401–418.

[49] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company, New York, 1949.

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